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ON THE FEEDBACK CONTROL OF THE WAVE EQUATION

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This paper addresses the problem of design of collocated and non-collocated controllers for a uniform bar without structural damping. The bar whose dynamics are described by the wave equation is required to perform a rest-to-rest maneuver. A time delay controller whose gains are determined using the root-locus technique is used to control the non-collocated system. The effect of sensor locations on the stability of the system is investigated when the actuator is located at the one end of the bar. The critical gains which correspond to a pair of poles entering the right-half of the s-plane and the optimal gains corresponding to locating the closed-loop poles at the left extreme of the root-locus for each vibration mode are determined. The gain which minimizes a quadratic cost, in the range of the critical gains, is selected as the optimum gain.

1. INTRODUCTION

The problem of determining stability of a collocated and non-collocated sensor–actuator pair is one of the current interest. Yang and Mote [1] have shown that instability is a potential problem when non-collocated control is used and propose an approach to stabilize flexible structures with non-collocated sensor–actuator pairs by introducing a time-delay element into the controller. In another paper, Yang and Mote [2] use their proposed approach to design a controller for active vibration control of an axially moving string. They show that by introducing a specific time delay into the controller structure, all the vibratory modes can be stabilized and the spillover instability can be avoided.

Chung and Tan [3] address the problem of control of the transverse vibration of an axially moving string using transfer function and wave cancellation concepts. The feedback controller is shown to consist of a velocity sensor, a proportional gain and a time delay. Datko *et al.* [4] investigate the effects of time delays in boundary feedback stabilization schemes for wave equations.

The location of actuators and sensors affects the transfer function zeros of a structure. Wie and Bryson [5] investigate the effect of actuator and sensor locations on the transfer function zeros using uniform bars and beams as generic models of flexible space structures. They also show that some flexible structures behave as if there were direct transmission between actuators and sensors (equal number of zeros and poles in the transfer function).

In this work, the closed-loop control of a uniform bar with boundary actuation whose equation of motion is the wave equation is studied. The locations of both zeros and poles in the open-loop transfer function are important for control design. Alli and Singh [6] use information about the poles of the open-loop transfer function of the wave equation to design the exact time-optimal controller. In this work, the same open-loop transfer function is used to design collocated and non-collocated feedback controllers. A time-delay proportional derivative controller is proposed to stabilize the closed-loop system. The effect of the magnitude of the gains of the controller on the stability of the collocated and non-collocated controller is studied using the root-locus technique. Equations are derived to determine the gains that correspond to the transfer of a pair of poles from the left half to the right half of the complex plane. Having determined the range of gains that result in stable controllers, the optimum gains which correspond to locating the poles at the left extreme of the root-loci for every mode are determined, which in conjunction with a quadratic cost, are used to select the optimum gain.

2. CONTROLLER DESIGN

A technique to design controllers for distributed parameter systems which uses the Lyapunov's direct method and a frequency domain approach is proposed. The proposed approach is exemplified by the design of a feedback controller for rest-to-rest control of an undamped rod in axial vibration. The control is applied at the boundary of the rod whose equation of motion is

$$\rho \frac{\partial^2 y(x,t)}{\partial t^2} = EA \frac{\partial^2 y(x,t)}{\partial x^2}, \tag{1}$$

where ρ is the mass per unit length, E the Young's modulus, A the cross-sectional area and L the length of the rod. The associated boundary conditions are

$$\frac{\partial y(0,t)}{\partial x} = -\frac{u(t)}{EA} \quad \text{and} \quad \frac{\partial y(L,t)}{\partial x} = 0 \tag{2}$$

and the initial and final conditions for a rest-to-rest maneuver are

$$y(x,0) = \frac{\partial y(x,0)}{\partial t} = \frac{\partial y(x,t_f)}{\partial t} = 0 \quad \text{and} \quad y(x,t_f) = y_f, \tag{3}$$

where t_f and y_f represent final time and displacement respectively.

2.1. LYAPUNOV-BASED CONTROLLER

Lyapunov's direct approach is proposed to arrive at a stable controller. A candidate Lyapunov function which includes terms which represent the kinetic, strain and pseudo-potential energy is

$$V(t) = \frac{1}{2}\int_0^L \rho \dot{y}(x,t)^2\,\mathrm{d}x + \frac{1}{2}\int_0^L EA(y'(x,t))^2\,\mathrm{d}x + \frac{1}{2}k_1(y(x_s,t) - y_f)^2, \tag{4}$$

where $(\cdot)$ and $(\cdot)'$ represent time and spatial derivatives, respectively, and x_s represents the location of the sensor. The time derivative of the candidate Lyapunov function leads to

$$\dot{V}(t) = \int_0^L \rho \dot{y}(x, t)\ddot{y}(x, t)\,\mathrm{d}x + \int_0^L EAy'(x, t)\dot{y}'(x, t)\,\mathrm{d}x + k_1\dot{y}(x_s, t)(y(x_s, t) - y_f). \tag{5}$$

Substituting equations (1) and (2) into equation (5) and after some algebraic manipulation, we have

$$\dot{V}(t) = \dot{y}(0, t)\,u(t) + k_1\dot{y}(x_s, t)(y(x_s, t) - y_f). \tag{6}$$

For the collocated sensor–actuator case, x_s will be zero. Substituting $x_s = 0$ into equation (6), we obtain

$$\dot{V}(t) = \dot{y}(0, t)[u(t) + k_1(y(0, t) - y_f)]. \tag{7}$$

The control law

$$u(t) = -\,k_2\dot{y}(0, t) - k_1(y(0, t) - y_f), \tag{8}$$

where $k_1, k_2 > 0$ leads to

$$\dot{V}(t) = -\,k_2\dot{y}(0, t)^2 \tag{9}$$

which is negative semi-definite and ensures that the system is stable.

For the non-collocated case, we have to have some relationship between the sensor and actuator position. Yang and Mote [1] proposed using a time delay in the control law to stabilize the system. We use this idea to design controllers for the non-collocated system undergoing rest-to-rest maneuvers in the frequency domain.

2.2. FREQUENCY DOMAIN APPROACH

Laplace transformation of equations (1) and (2) leads to

$$y''(x, s) - c^2s^2y(x, s) = 0, \tag{10}$$

where $1/c$ is the wave speed and c is defined as

$$c^2 = \rho/EA \tag{11}$$

and the Laplace transformation of the boundary conditions are

$$y'(0, s) = -\,u(s)/EA \quad \text{and} \quad y'(L, s) = 0. \tag{12}$$

The solution of equation (10) is

$$y(x, s) = C_1 \cosh(csx) + C_2 \sinh(csx), \tag{13}$$

where the constants C_1 and C_2 are determined by evaluating equation (13) at the boundaries. The resulting open-loop transfer function is

$$\frac{y(x, s)}{u(s)} = \frac{\cosh(cs(L - x))}{EA\, cs \sinh(csL)} \tag{14}$$

with the system open-loop poles located at

$$s = 0, 0 \quad \text{and} \quad s = \pm \frac{n\pi}{cL} i, \quad n = 1, 2, \ldots, \infty. \tag{15}$$

Consider a output feedback controller illustrated by Figure 1, where the system transfer function has been rewritten as

$$\frac{y(x, s)}{u(s)} = \frac{Z_0(x, s)}{P_0(s)}, \tag{16}$$

where

$$Z_0 = \cosh(cs(L - x)) \quad \text{and} \quad P_0 = \mathrm{EAc} \sinh(csL). \tag{17}$$

The control input to the plant is given by the equation

$$u(s) = r(s) - y(x_s, s)G(s), \tag{18}$$

where $y(x_s, s)$ is the displacement measured by a sensor located at x_s.
Substituting equation (18) and $x = x_s$ into equation (16), we have

$$y(x_s, s) = \frac{Z_0(x_s, s)r(s)}{P_0(s) + Z_0(x_s, s)G(s)}. \tag{19}$$

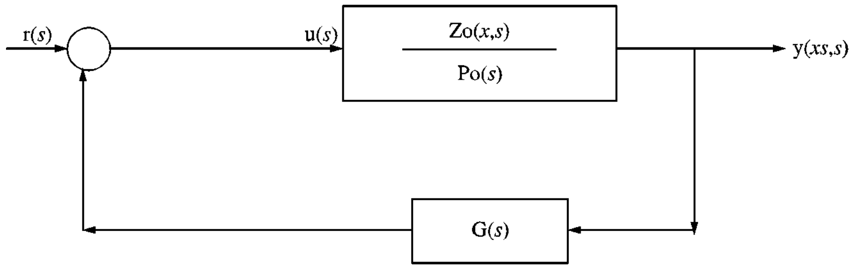

Figure 1. Feedback structure of the controller.

Substituting equations (18) and (19) into equation (16) and rearranging the resulting equation, we have

$$y(x, s) = \frac{Z_0(x, s)}{P_0(s)}\left[r(s) - \frac{Z_0(x_s, t)G(s)r(s)}{P_0(s) + Z_0(x_s, s)G(s)} \right] \tag{20}$$

yielding

$$\frac{y(x, s)}{r(s)} = \frac{Z_0(x, s)}{P_0(s) + Z_0(x_s, s)\, G(s)} \tag{21}$$

which is the closed-loop transfer function. The closed-loop characteristic equation is

$$g(s) = P_0(s) + Z_0(x_s, t)G(s) = 0 \tag{22}$$

which will be used to study the stability of feedback controllers.

2.2.1. *Collocated sensor–actuator case*

For the collocated sensor–actuator case, $x_s = 0$ and as indicated in equation (8), the Lyapunov stable control law leads to

$$G(s) = k_1(ks + 1), \tag{23}$$

where $k = k_2/k_1$, the gain ratio. Substituting equation (23) and $x_s = 0$ into equation (22), the closed-loop characteristic equation is

$$g(s) = EAcs \sinh(csL) + k_1(ks + 1)\cosh(csL) = 0. \tag{24}$$

We will use the root-locus technique to study the effects of the gains k_1 and k.

Pan and Chao [8], propose a technique to generate the root-loci which can be easily used to generate root-loci of distributed parameter systems. For a system with the characteristic equation

$$g(s) = B(s) + KA(s) = 0 \tag{25}$$

they suggest defining differential equations

$$\mathrm{d}K/\mathrm{d}t = \pm 1, \tag{26}$$

where t is a dummy variable and

$$\mathrm{d}g/\mathrm{d}t = -g \tag{27}$$

which implies that g is stable. Using the chain rule, we have

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial K}\frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\partial g}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}t} = -g \tag{28}$$

which can be rewritten as

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{-(g \pm \partial g/\partial k)}{\partial g/\partial s}, \tag{29}$$

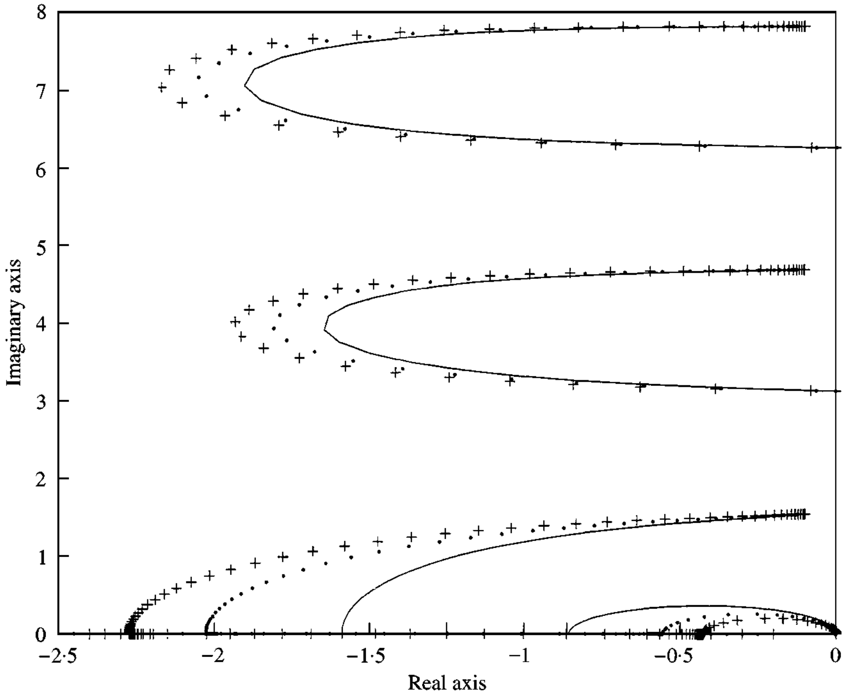

Figure 2. Root locus for the collocated case. $+++$, $k=5$; …, $k=4$; $-$, $k=3$.

where $K(0)=0$, which corresponds to the open-loop case. The corresponding initial condition for equation (29) is $s(0)=S_0$, where S_0 is the set of all the open-loop poles, each of which leads to a root locus. Equation (29) is used to generate root loci where $K=k_1$ and k is fixed. Figure 2 represents the root loci for different values of k. It is evident that one pole approaches the leftmost breakoff point from ∞. Wei and Bryson [5], conjecture that this is attributable to direct transmission from actuator to sensor. Figure 2 corroborates the convergence of the root loci to the zeros which are located at

$$s=-\frac{1}{k} \quad \text{and} \quad s=\frac{2n-1}{2}\pi i, \quad n=1,2,\ldots,\infty \tag{30}$$

assuming EA, c and L are equal to unity. It is evident from the root locus that they do not cross the imaginary axis which agrees with the stability of the Lyapunov-based controller which required the feedback gains to be positive.

The time response of the rest-to-rest maneuver is presented in Figure 3. Figure 2 show that the system with the collocated sensor–actuator and PDD controller is stable for any $k_1, k>0$.

2.2.2. *Non-collocated sensor–actuator case*

Practical limitation might not permit collocation of sensor–actuator pairs. On the other hand, non-collocated sensor–actuators have the potential of destabilizing the system for some feedback gains. In this section, we attempt to obtain a relationship between the sensor and the actuator position to make the system stable using the root-locus technique.

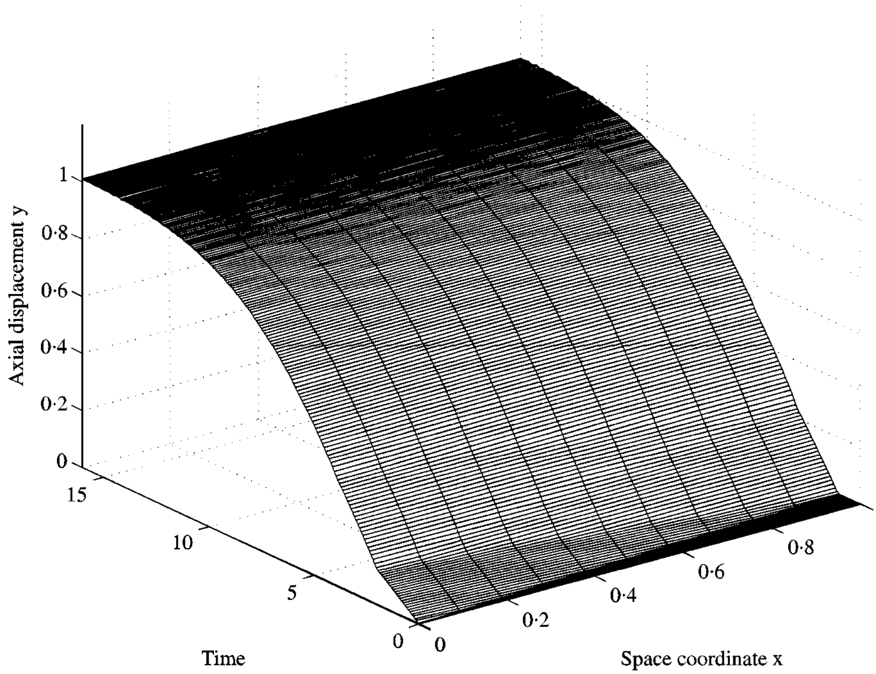

Figure 3. Time response of the collocated case.

The open-loop poles of the undamped wave equation are located on the imaginary axis. We need to determine the direction of motion of the root loci from these open-loop poles. If

$$\mathrm{d}\sigma_n/\mathrm{d}k_1 < 0, \tag{31}$$

where

$$s_n = \sigma_n + \mathrm{i}\omega_n \tag{32}$$

and σ_n is the real portion of the closed-loop poles, then the system is stable for $k_1 \varepsilon (0, k_{1cr})$. k_{1cr} is the smallest gain that can destabilize the system. The derivative of equation (32) w.r.t. k_1 is

$$\frac{\mathrm{d}s_n}{\mathrm{d}k_1} = \frac{\mathrm{d}\sigma_n}{\mathrm{d}k_1} + \frac{\mathrm{d}\omega_n}{\mathrm{d}k_1}\mathrm{i}, \tag{33}$$

where $\mathrm{d}\sigma_n/\mathrm{d}k_1$ is the real portion of equation (33). If real $(\mathrm{d}s_n/\mathrm{d}k_1)$ is negative when the gain k_1 changes form zero to zero plus ($k_1 = 0 \to 0^+$), then the system is stable for some range of k_1.

To obtain $\mathrm{d}s_n/\mathrm{d}k_1$, we study equation (22) with a PD controller

$$g(s) = EAcs\sinh(csL) + k_1(ks + 1)\cosh(cs(L - x_s)) = 0 \tag{34}$$

which is rewritten as

$$k_1 = \frac{-EAcs\sinh(csL)}{(ks+1)\cosh(cs(L-x_s))}. \tag{35}$$

The derivative of equation (35) w.r.t. s_n leads to

$$\frac{\mathrm{d}k_1}{\mathrm{d}s_n} = \frac{-EAc\,[\sinh(csL) + cLs\cosh(cs(L-x_s))](ks+1)\cosh(cs(L-x_s))}{(ks+1)^2(\cosh(cs(L-x_s)))^2}$$
$$+\frac{EAcs\,[\sinh(csL)[k\cosh(cs(L-x_s))(ks+1)c(L-x_s)\sinh(cs(L-x_s))]}{(ks+1)^2(\cosh(cs(L-x_s)))^2}. \tag{36}$$

After some simplification and substituting $s_n = (n\pi/cL)\mathrm{i}$, which are the open-loop poles, into equation (36), we obtain

$$Re\left(\frac{\mathrm{d}k_1}{\mathrm{d}s_n}\right) = \frac{\mathrm{d}k_1}{\mathrm{d}\sigma_n} = -4EAc\left[\cos\left(\frac{n\pi x_s}{L}\right) + \sin\left(\frac{n\pi x_s}{L}\right)\right]. \tag{37}$$

Rearranging equation (37), we have

$$\frac{\mathrm{d}\sigma_n}{\mathrm{d}k_1} = \frac{-1}{4EAc[\cos(n\pi x_s/L) + \sin(n\pi x_s/L)]} = \frac{-1}{4EAc[\sqrt{2}\sin(n\pi x_s/L + \pi/4)]} \tag{38}$$

which cannot be negative for all values of n except when $x_s = 0$ which corresponds to the collocated case. This proves that a PD controller will not stabilize a non-collocated controller. Since it is intuitive that there should exist a time delay that relates the signal at the location of the actuator and the sensor, we propose a time-delay feedback controller of the form

$$G(s) = k_1 \mathrm{e}^{-T_d s}(ks+1) = 0, \tag{39}$$

where T_d is the time delay. The resulting closed-loop characteristic equation is

$$g(s) = EAcs\sinh(csL) + k_1\mathrm{e}^{-T_d s}(ks+1)\cosh(cs(L-x_s)) = 0. \tag{40}$$

After some simplification of equation (40), k_1 can be represented as

$$k_1 = \frac{-EAcs(1-\mathrm{e}^{-2cLs})}{(\mathrm{e}^{-s(cx_s+T_d)} + \mathrm{e}^{-s(c(2L+x_s)+T_d)})(ks+1)}. \tag{41}$$

We now determine the relationship between the time delay and the location of the sensor by using the procedure described above to determine stability. This results in the equation

$$\frac{\mathrm{d}\sigma_n}{\mathrm{d}k_1} = \left(\mathrm{Re}\left(\frac{\mathrm{d}s_n}{\mathrm{d}k_1}\right)\right)_{s=0} = -1 \tag{42}$$

for the poles located at $s = 0$ when k_1 changes 0 to 0^+ and

$$\frac{\mathrm{d}\sigma_n}{\mathrm{d}k_1} = \left(\mathrm{Re}\left(\frac{\mathrm{d}s_n}{\mathrm{d}k_1}\right)\right)_{s_n} = (n\pi/c\mathrm{L})\mathrm{i} = -4EAc\left[\cos\left(\frac{n\pi}{cL}(cx_s + T_d)\right) + \sin\left(\frac{n\pi}{cL}(cx_s + T_d)\right)\right]$$

$$= -4EAc\left[\sqrt{2}\sin\left(\frac{n\pi}{cL}(cx_s + T_d) + \frac{\pi}{4}\right)\right] \tag{43}$$

for the poles located at $s = \pm n\pi\mathrm{i}/cL$. Because of stability requirement, $\mathrm{d}\sigma_n/\mathrm{d}k_1$ should always be less than zero, which requires

$$2mn\pi \leqslant \frac{n\pi}{cL}(cx_s + T_d) + \frac{\pi}{4} \leqslant (2mn + 1)\pi. \tag{44}$$

Since $n = 1, 2, \ldots, \infty$ and T_d has to be positive, the above inequality is satisfied only when

$$cx_s + T_d = 2mcL, \quad m = \{1, 2, \ldots, \infty\} \tag{45}$$

which requires

$$T_d = c(2mL - x_s) \quad \text{for one } m = \{1, 2, \ldots, \infty\}. \tag{46}$$

Equations (46) shows that the time delay is a function of the wave speed, length of the bar and the sensor position. The range of the feedback gains for which the time-delay controller is stable has not yet been determined. Therefore, to determine the range of gains for which the time-delay controller is stable, we need to determine the smallest gain which corresponds to the intersection of a root locus with the imaginary axis. The procedure to achieve this is described in the next section.

2.2.3. *Determination of destabilizing gain*

In order to obtain the maximum gains (or critical gain), we have to find the inter-section of the root loci with the imaginary axis at points other than the open-loop poles. Let the intersection be

$$s = \pm \mathrm{i}\alpha_n, \tag{47}$$

where α_n is the location on the imaginary axis which corresponds to the intersection of the nth root locus. Substituting equations (46) and (47) into equation (41) and equating the imaginary part of the right-hand side of equation (41) to zero, since k_1 is a real number, we arrive at the equation

$$k\alpha_n(-\sin(2(m-1)cL\alpha_n) + \sin(2mcL\alpha_n)) - \cos(2m-1)cL\alpha_n) + \cos(2mcL\alpha_n) = 0,$$

$$n, m = 1, 2, \ldots, \infty \tag{48}$$

which can be solved to determine α_n. The real part of equation (41) gives us the gains which corresponds to the point of intersection of the root locus with the imaginary axis:

$$(k_1)_n = \frac{EAc\alpha_n}{2}(-\sin(2(m-1)cL\alpha_n) + \sin(2mcL\alpha_n)), \quad n, m = 1, 2, \ldots, \infty \tag{49}$$

Therefore, for stability, the gain k_1 should lie in the range $(0, \min((k_1)_n)$.

If we locate the sensor at $x_s = L$, equation (41) simplifies to

$$k_1 = \frac{EAcs\sinh(cLs)}{(ks+1)\mathrm{e}^{-2mcLs}}, \quad m = 1, 2, \ldots, \infty \tag{50}$$

which has to be satisfied for one m. Following the procedure delineated above, the location of the root loci with the imaginary axis is given by the solutions of the equation

$$\tan(cL\alpha_n) - k\alpha_n = 0, \quad n = 1, 2, \ldots, \infty \tag{51}$$

and the corresponding gain is

$$(k_1)_n = EAc\frac{k\alpha_n^2}{1 + (k\alpha_n)^2} = EAc\frac{1}{1/k\alpha_n^2 + k}, \quad n = 1, 2, \ldots, \infty. \tag{52}$$

We note that as α_n tends to ∞, $(k_1)_n$ tends to

$$\lim_{\alpha_n \to \infty} (k_1)_n = \frac{EAc}{k}, \quad n = 1, 2, \ldots, \infty. \tag{53}$$

Thus, the critical gain is given by the equation

$$(k_1)_{critical} = \min(k_1)_n = EAc\frac{1}{1/k(\min\alpha_n)^2 + k}, \quad n = 1, 2, \ldots, \infty. \tag{54}$$

Figure 4 illustrates the variation of the critical gain as a function of the location of intersection of the root locus with the imaginary axis. It is evident from the figure that the gain k_1 asymptotically approaches the value EAc/k and that the critical gain for the systems is the gain corresponding to the root loci which initiate at the location of the rigid body poles.

Figure 5 illustrates various root-loci with $x_s = L$, for different value of gain ratio k, Figure 6 is a three-dimensional plot which shows the time response of different points of the bar, when the sensor is located at $x_s = L$ and the controller is a time-delay proportional plus derivative feedback controller.

For non-collocated case, the sensor position should be such that x_s/L is an irrational number to ensure that all the modes are observable. Consider the case when $x_s = L/2$, if we substitute $x_s = L/2$ into the open-loop transfer function, we have

$$\frac{y(x_s, s)}{u(s)} = \frac{\cosh(cs(L/2))}{EAcs\sinh(csL)} = \frac{\cosh(cs(L/2))}{EAcs\,2\cosh(cs(L/2), \sinh(cs(L/2))} = \frac{1}{EAcs\,2\sinh(cs(L/2))} \tag{55}$$

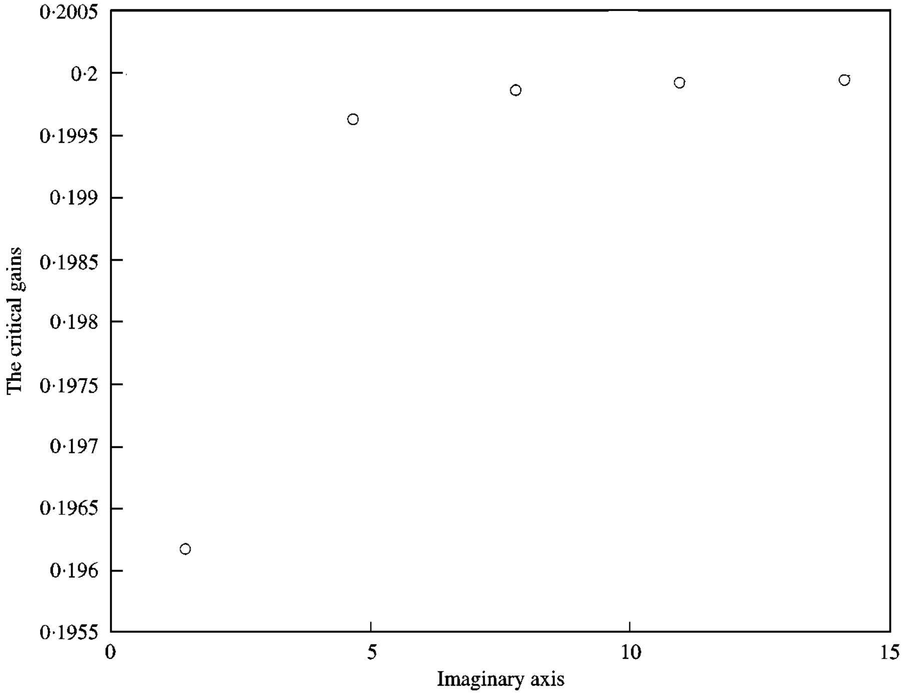

Figure 4. Critical gain for the non-collocated case ($x_s = L$), for $k = 5$.

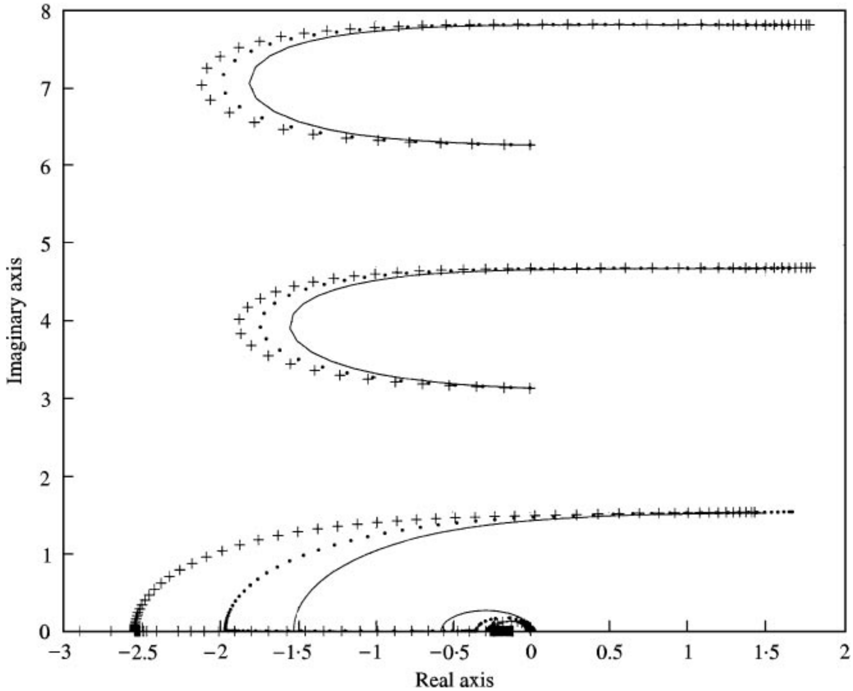

Figure 5. Root locus for $x_s = L$. $+++$, $k = 9$; $\ldots, k = 7$; $-$, $k = 5$.

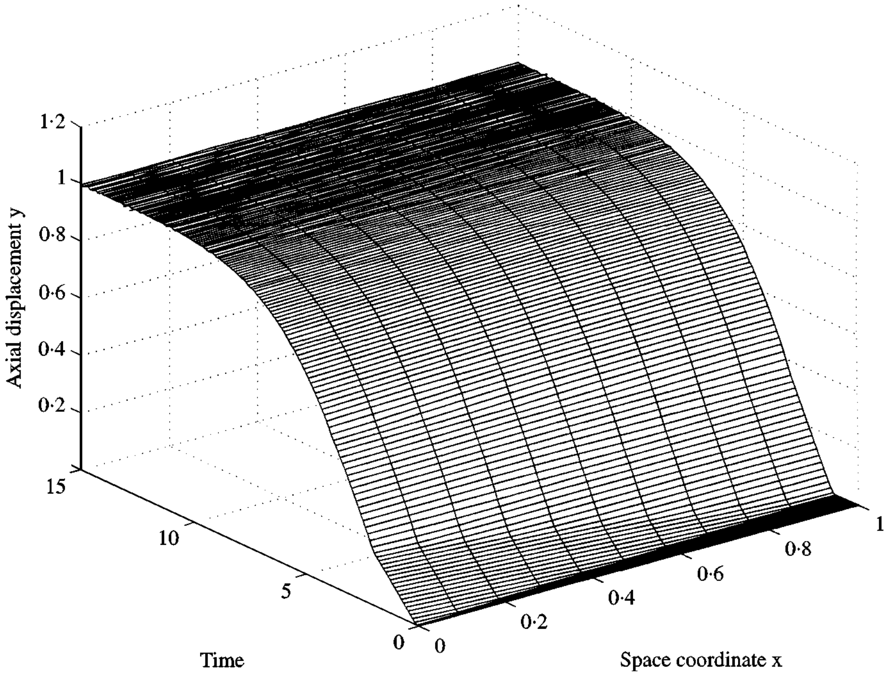

Figure 6. Time response of the non-collocate case ($x_s = L$).

which indicates pole-zero cancellation resulting in only the poles at $(2n\pi/cL)\,\mathrm{i}$ being observable. Figure 7, which plots the root locus also indicates that the poles at $((2n-1)\pi/cL)\,\mathrm{i}$ are unobservable. Figure 8 shows that the time response of this system results in residual energy which can be attributed to spillover of the control into modes that cannot be observed.

2.2.4. *Optimum overall gain*

It is evident from the root loci in Figure 5 that the slope of the root locus goes to infinity at points where the root locus has reached the farthest point in the left-half of the complex plane. Substitute $s = \sigma + \mathrm{i}\omega$ into equation (41) and taking its derivative with respect to the real part of the poles, we have

$$\left(\frac{\mathrm{d}k_1}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^{-1} = 0 \tag{56}$$

when the slope of the root locus is infinity.

Figure 9 illustrates the gain which corresponds to when the root locus changes direction from moving to the left to moving to the right of the complex plane. It can be seen that the largest gain corresponds to the rigid-body mode by which time the rest of the roots have started moving toward the imaginary axis. The gains asymptotically tend to a constant for root-loci of the higher modes. The roots of equation (56) give only the optimum gain for each mode, however we need the overall optimum gain for the system which generate the quickest response.

The cost function V

$$V = \int_0^{\infty} (y_f - y(L, t))^2\,\mathrm{d}t \tag{57}$$

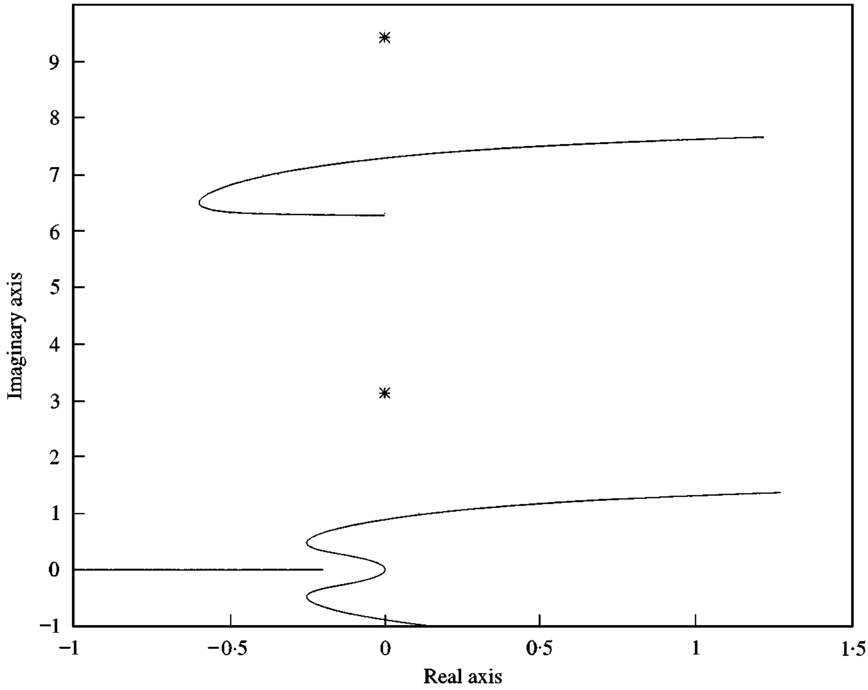

Figure 7. Root-locus of the non-collocated case ($x_s = L/2$).

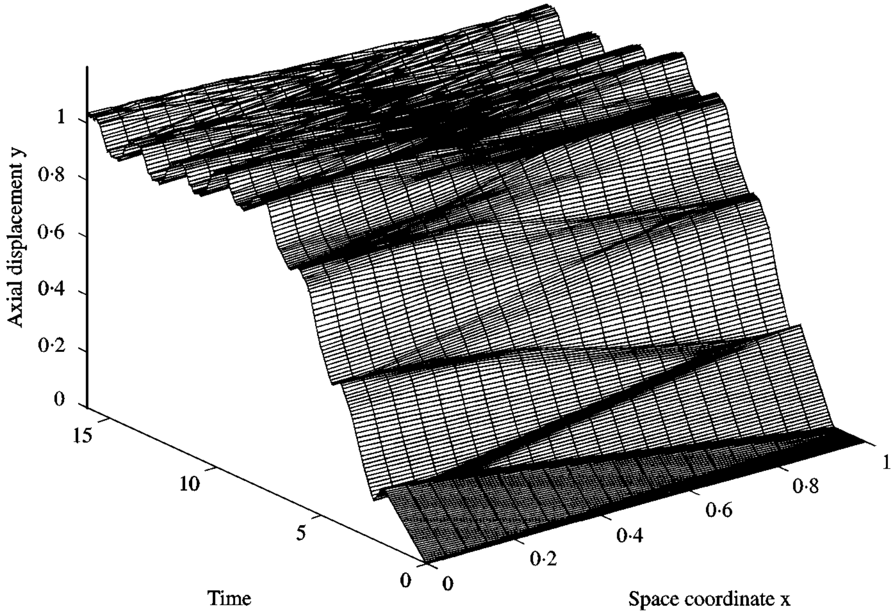

Figure 8. Time response of the non-collocated case ($x_s = L/2$).

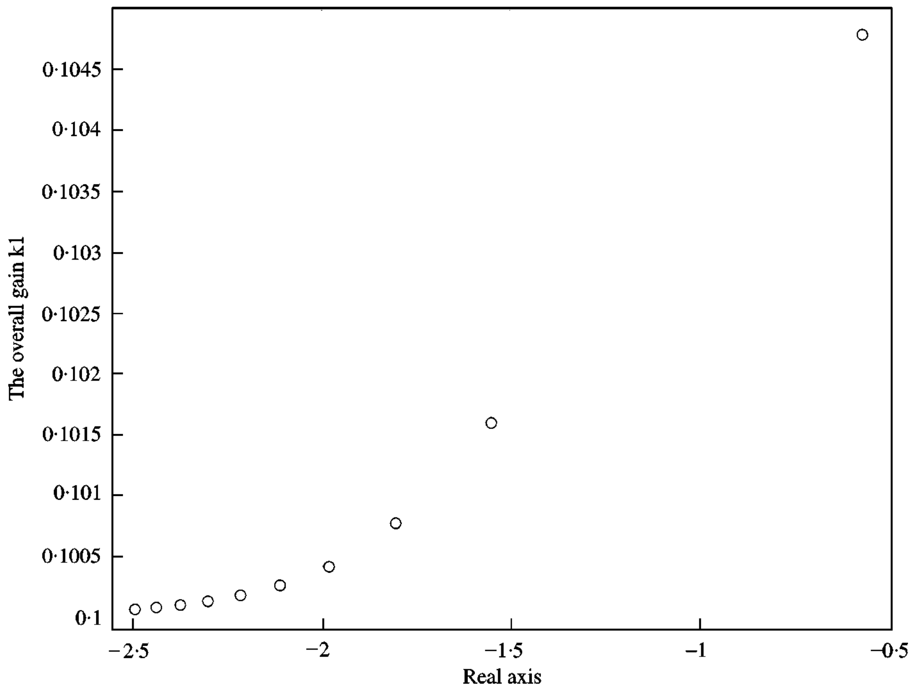

Figure 9. Optimum overall gain of the non-collocated case ($x_s = L$) for $k = 5$.

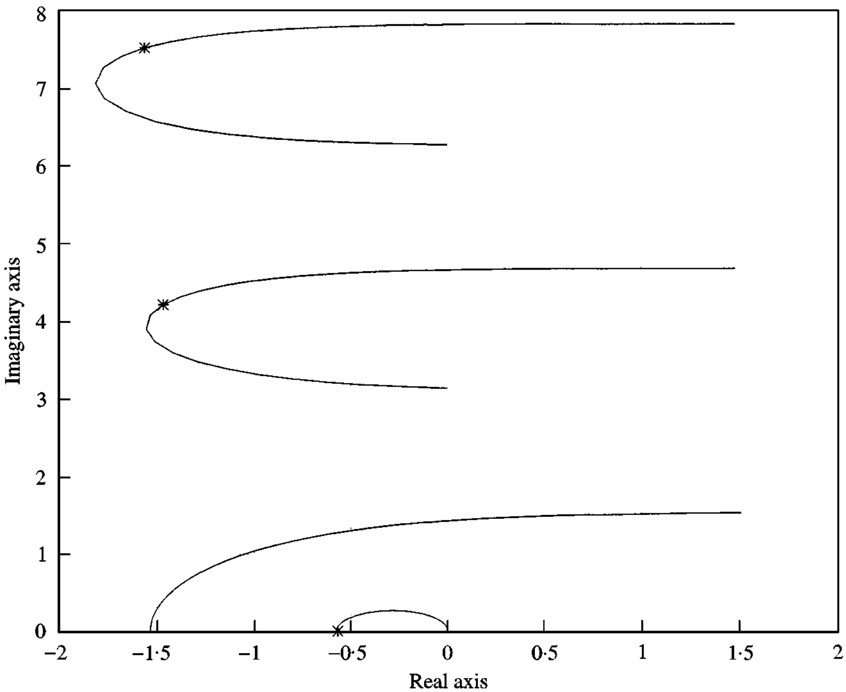

Figure 10. Poles corresponding the optimum overall gain ($x_s = L$). *For the optimum gain.

is used to determine the optimum gain in the range where the system poles are at the extreme left of the root loci (Figure 9). Table 1 lists the cost for a range of the gain k_1.

Figure 10 illustrates the pole locations when we apply the gain $k_1 = 0{\cdot}1048$ which corresponds to the smallest cost in the range of interest.

To compare the system time response for different overall gains, we simulate the system for $k = 5$. In Figure 11, it is evident that the system response is similar to a critically damped system when we use the optimum gain obtained from Table 1.

3. CONCLUSIONS

A Lyapunov-based approach has been used to design controllers for the wave equation with collocated sensor–actuator pairs and a frequency domain approach is used to study

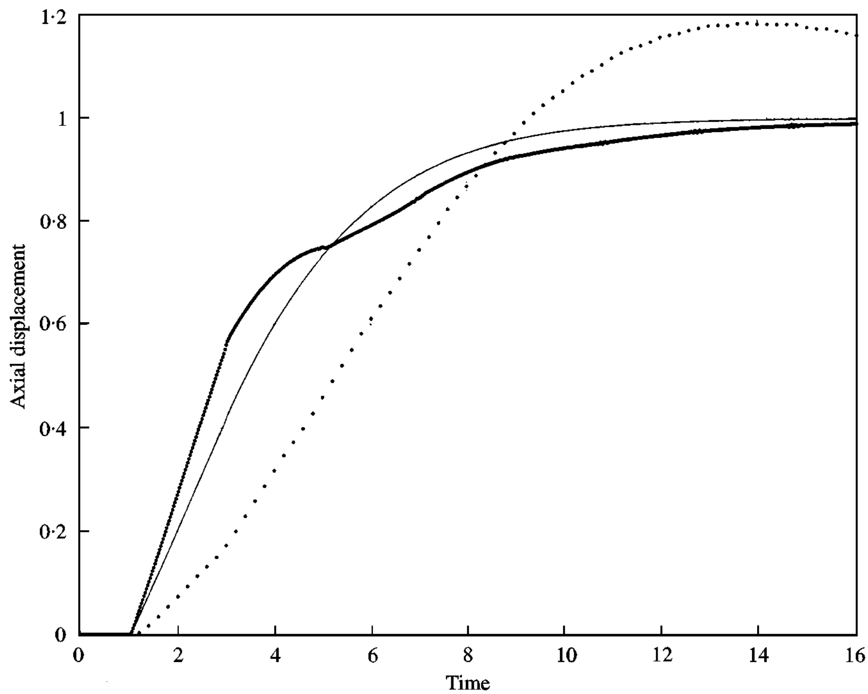

Figure 11. Time response corresponding the optimum overall gain ($x = L$). dashed line. k1 = 0·05; solid line, k1 = 0·10479 (opt): bold solid line. k1 = 0·14.

TABLE 1

Cost versus feedback gain

k_1	total cost
0·1000	2·772314
0·1015	2·754139
0·1030	2·736514
0·1045	2·719417
0·1048	2·716059

the non-collocated case. The root locus is used to determine the gains of a time-delay proportional and derivative feedback controller which stabilize the system. Based on the root-locus analysis, we make the following conclusions: For the collocated case any gain greater than zero makes the system stable. For the non-collocated case, there exists a specific time delay which is a function of wave speed, bar length and sensor position which can stabilize the system provided the ratio of the sensor location to the length of the bar is an irrational number or the sensor is located at the boundary. The range of stable gains is determined by determining the minimum gain that forces a root locus into the right half of the complex plane. A technique to determine the optimum gain which results in the close-loop poles being located at the left extreme of the root locus has been proposed, which is used to select a gain which minimizes a quadratic cost.

REFERENCES

1. B. Yang and C. D. Mote 1992 On time delay in non-collocated control of flexible mechanical systems. *Journal of Dynamics Systems, Measurement and Control* **114**, 409–415.
2. B. Yang and C. D. Mote 1991 *Journal of Applied Mechanics* **58**, 189–196. Active vibration control of the axially moving string in the S domain.
3. C. H. Chung and C. A. Tan 1993 *ASME Vibration and Control of Mechanical Systems* **DE-61**, 21–28. Active vibration control of the axially moving string by cancellation.
4. R. Datko, J. Lagnese and M. P. Polis 1986 *Siam Journal on Control and Optimization* **24**, 152–155. An example on the effect of time delays in boundary feedback stabilization of wave equation.
5. B. Wie and R. L. Bryson 1981 *Proceedings of the Third Conference on Dynamics and Control of Large Structures*, Blacksburg, Virginia, 153–174. Modelling and control of flexible space structure.
6. H. Alli and T. Singh 1995 *AIAA Guidance, Navigation and Control Conference*, Baltimore, MD, 102–111. Exact time optimal control of the wave equation.
7. J. L. Junkins and Y. Kim 1993 *Introduction to Dynamics and Control of Flexible Structures.* AIAA Education Series.
8. C. T. Pan and K. S. Chao 1978 *IEEE Transactions on Automatic Control* **AC-23**, 856–860. A computer-aided root-locus method.