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Desensitized control of vibratory systems with friction: linear programming approach

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SUMMARY

This paper presents a linear programming based approach for the design of controllers for rest-to-rest manoeuvres of vibratory systems subject to friction. A multiple spring mass system is considered in the development where friction and control input forces act on the same mass. The result of the linear programming is a control profile for rest-to-rest manoeuvres where the static and Coulomb friction is included in the system model. The positive pulse controller is also developed when the available frictional force brings the system to rest. These controllers can be applied to precision positioning systems and servo applications where the effect of friction and flexibility are significant. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: linear programming; friction; vibratory system; pulse width control

1. INTRODUCTION

Friction is a ubiquitous non-linearity encountered in mechanical systems. Various compensation techniques are available for regulating frictional systems such as adaptive friction feedforward/feedback and impulsive control [1, 2]. The stick-slip effect is accentuated when regulating vibratory systems which are characterized by velocity reversals around the small velocity region. One way of keeping the system stick-slip free is to maintain positive velocity of the frictional body during the manoeuvre. This positive velocity constraint of the frictional body creates a constant Coulomb friction force, which is a bias force term added to the input force. With this constraint, optimal control design techniques for linear systems can be used for systems with friction. One method to find a time optimal control profile is input shaping. Input shaping is a technique to eliminate vibration of a system by pre-shaping the control input to cancel the flexible dynamics of the system. Input shaping is developed for practical applications by Singer and Seering [3], where sequence of impulses are introduced to shape the control input. Since

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then, input shaping has been successfully adopted for many vibration control problems. The input shaping technique can also be used to design saturating controllers such as time-optimal control, fuel/time optimal control, and desensitized control [4, 5]. Recently, the input shaping technique has been extended to account for the Coulomb friction in the system [6, 7]. Another novel approach to solve the rest-to-rest manoeuvre problem of the vibratory system is linear programming [8, 9]. Although the accuracy of the control profile is limited by the number of samples and convergence tolerance, linear programming guarantees generation of a near globally optimal control profile. The velocity constraint of the frictional body can easily be included in the problem formulation in order to make the system linear. If both positive and negative velocities of the frictional body are considered, mixed integer programming is required [10]. Since the result of the linear programming problem is a feed-forward controller, desensitization to the parametric uncertainty is required for practical applications. With the linear programming formulation, the parametric uncertainty can be handled by defining the sensitivity states [11]. The system model used in the controller design can represent many applications such as hard disk drives and flexible link robots where friction is present at the actuator and the end effector position needs to be regulated.

2. PROBLEM FORMULATION

An n-spring-mass-damper system under friction is illustrated in Figure 1. The control and friction forces are collocated and act on the sth mass. Since the non-collocated problem requires switching of models as a function of stiction, the problem is not amenable to solution using the linear programming approach. The design goal is to determine a control input which moves the system from the initial states to the specified final states. By defining $q \equiv [x_1 x_2 \cdots x_s \cdots x_n]^{\mathrm{T}}$, the equation of motion of the system in Figure 1 can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{D}u - \mathbf{D}f(u, \mathbf{q}) \tag{1}$$

where, M, C, and K are mass, damping, and stiffness matrices respectively. Since, it is a single input system, the control influence vector D has a non-zero value only in the sth row. The friction force f is represented by the classic static and Coulomb friction model, which is

$$f = \begin{cases} f_{\mathrm{c}} \ \mathrm{sgn}(\dot{x}_s) & \text{if } \dot{x}_s \neq 0 \\ f_{\mathrm{s}} \ \mathrm{sgn}(u_s) & \text{if } \dot{x}_s = 0 \text{ and } u_s > f_{\mathrm{s}} \\ u_s & \text{if } \dot{x}_s = 0 \text{ and } u_s \leqslant f_{\mathrm{s}} \end{cases} \tag{2}$$

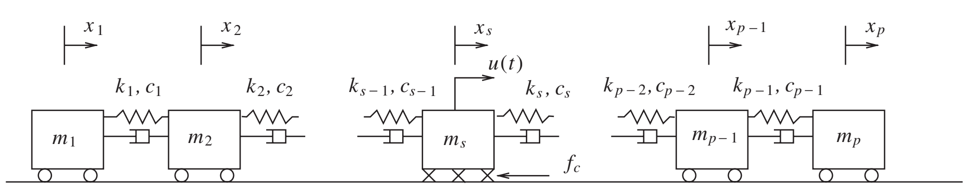

Figure 1. Flexible system with friction.

where, f_c is Coulomb friction and f_s is static friction. u_s is the sum of control input and spring force applied to the sth mass, which is

$$u_s = u + k_s(x_{s+1} - x_s) - k_{s-1}(x_s - x_{s-1}) \tag{3}$$

If the velocity of the sth mass is constrained to be positive during the manoeuvre, the friction force becomes $f(u, \mathbf{q}) = f_c$. With this friction force, the system is linear with a biased input $u - f_c$. When linear programming is used to determine the control input, additional constraints of positive velocity of the sth mass at each sampling time should be imposed. The discrete time state space representation of the system can now be written as

$$\mathbf{x}_{k+1} = \mathbf{A_d}\mathbf{x}_k + \mathbf{B_d}(u_k - f_c) \tag{4}$$

where, $\mathbf{x} = [\mathbf{q}^T \dot{\mathbf{q}}^T]^T$. The initial and final states are given as

$$\mathbf{x}_1 = \mathbf{x}(0), \quad \mathbf{x}_{N+1} = \mathbf{x}(t_f) \tag{5}$$

where, t_f is the final time and N is the number of samples in the interval $[0 \quad t_f]$. The linear programming problem is to find a minimum t_f while satisfying the system equation with initial/final state constraints, velocity constraints and input constraints, which is similar to the approach by Driessen [8]. The state vector at the final time can be computed from the initial condition and control history resulting in the constraint

$$\mathbf{x}_{N+1} = \mathbf{A_d}^N \mathbf{x}_1 + \sum_{i=1}^{N} \mathbf{A_d}^{N-i}\mathbf{B_d}u_i - \sum_{i=1}^{N} \mathbf{A_d}^{N-i}\mathbf{B_d} f_c \tag{6}$$

By rewriting Equation (6), the final state equality constraints become

$$[\mathbf{A_d}^{N-1}\mathbf{B_d} \quad \mathbf{A_d}^{N-2}\mathbf{B_d} \quad \dots \quad \mathbf{A_d}\mathbf{B_d} \quad \mathbf{B_d}] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \mathbf{x}_{N+1} - \mathbf{A_d}^N \mathbf{x}_1 + \sum_{i=1}^{N} \mathbf{A_d}^{N-i}\mathbf{B_d} f_c \tag{7}$$

Positive velocity constraints on the motion of the sth mass, at each sampling instant yields inequality constraints. Define a output vector C_d, the velocity of the sth mass becomes

$$\dot{x}_{s,k+1} = \mathbf{C_d}\mathbf{A_d}^k \mathbf{x}_1 + \sum_{i=1}^{k} \mathbf{C_d}\mathbf{A_d}^{\mathbf{k-i}}\mathbf{B_d}u_i - \sum_{i=1}^{k} \mathbf{C_d}\mathbf{A_d}^{k-i}\mathbf{B_d} f_c \tag{8}$$

where, $k = 1, 2, \dots, N$. The inequality constraints are

$$\dot{x}_{s,k+1} > 0 \quad (k = 1, 2, \dots, N) \tag{9}$$

In order to form a standard linear programming problem, Equation (9) can be written as

$$\dot{x}_{s,k+1} \geqslant \varepsilon \qquad (k = 1, 2, \ldots, N) \tag{10}$$

where, ε is a small positive number. Another constraint in the design is that the first input should be greater than the static friction in order to start the manoeuvre. That is

$$u_1 \geqslant f_s + \varepsilon \tag{11}$$

Inequality constraints (Equation (10) and(11)) can be represented in a matrix form as

$$\begin{bmatrix} -1 & 0 & \cdots & 0 & 0 \\ -\mathbf{C_d B_d} & 0 & \cdots & 0 & 0 \\ -\mathbf{C_d A_d B_d} & -\mathbf{C_d B_d} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -\mathbf{C_d A_d^{N-2} B_d} & -\mathbf{C_d A_d^{N-3} B_d} & \cdots & -\mathbf{C_d B_d} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{bmatrix}$$

$$\leqslant \begin{bmatrix} -f_s - \varepsilon \\ \mathbf{C_d A_d x}_1 - \mathbf{C_d B_d}\, f_c - \varepsilon \\ \mathbf{C_d A_d^2 x}_1 - \mathbf{C_d A_d B_d} f_c - \mathbf{C_d B_d}\, f_c - \varepsilon \\ \vdots \\ \mathbf{C_d A_d^{N-1} x}_1 - \sum_{i=1}^{N-1} \mathbf{C_d A_d^{N-1-i} B_d} f_c - \varepsilon \end{bmatrix} \tag{12}$$

The control input has lower and upper bounds such that

$$-u^p \leqslant u_k \leqslant u^p \qquad (k = 1, 2, \ldots, N) \tag{13}$$

where u^p is the maximum control input value.

3. NUMERICAL EXAMPLES

A two-mass-spring system illustrated in Figure 2 is considered to exemplify the proposed technique. When the first mass velocity is maintained positive during the manoeuvre, the

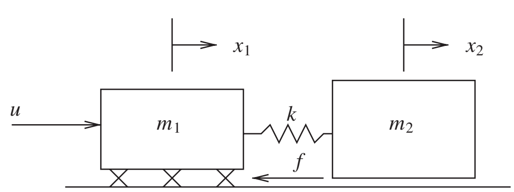

Figure 2. Floating oscillator with friction.

equation of the motion is written as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}(u - f_c) \tag{14}$$

The parameter values used in the simulation are chosen as $m_1 = 80$ kg, $m_2 = 100$ kg, $k = 111111.1111$ kg/s^2, $f_s = 137$ N, $f_c = 111$ N, and $u^p = 500$ N. The first simulation is performed with the initial and final states of $\mathbf{x}(0) = [0\ 0\ 0\ 0]^T$ and $\mathbf{x}(t_f) = [0.1\ 0.1\ 0\ 0]^T$. The control input history and its discrete time linear simulation response are shown in Figure 3. The velocity of the first mass plotted with a solid line in Figure 3 remains positive during the manoeuvre. Therefore, the corresponding control solution is a near time-optimal bang–bang profile with three switches.

The second simulation is performed with the initial and final states of $\mathbf{x}(0) = [0\ 0\ 0\ 0]^T$ and $\mathbf{x}(t_f) = [0.001\ 0.001\ 0\ 0]^T$. The corresponding control input history and its response are shown in Figure 4. The response plot in Figure 4 shows that the velocity of the first mass stays very close to zero during the manoeuvre for some time. In fact, the velocity of the first mass stays zero for some time if ε in Equation (10) approaches zero. In Figure 4, the controller is trying to compensate for the Coulomb friction and spring force during the stiction period in order to maintain stiction of the first mass. However, Coulomb friction force disappears when the first mass is stuck. Therefore, the controller does not need to compensate Coulomb friction when the first mass velocity is zero. Therefore, it is necessary to subtract the Coulomb friction value from the resulting control. In addition, if the spring force is not large enough to overcome static friction, the first mass will stay stuck. This condition leads to an equivalent control, u_{eq},

$$u_{eq,k} = 0 \quad \text{if } f_c - f_s < u_k < f_c + f_s \quad \text{and} \quad \dot{x}_{1,k} = 0 \tag{15}$$

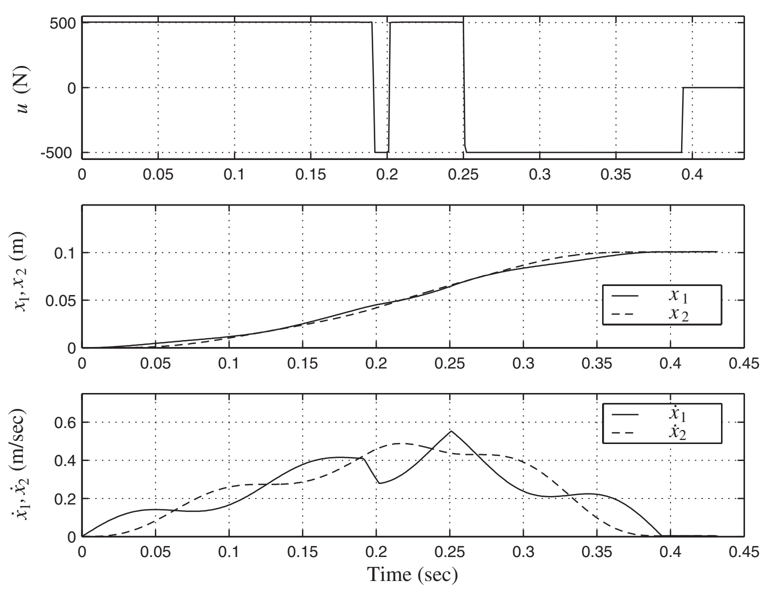

Figure 3. Control input and responses, displacement 0.1 m.

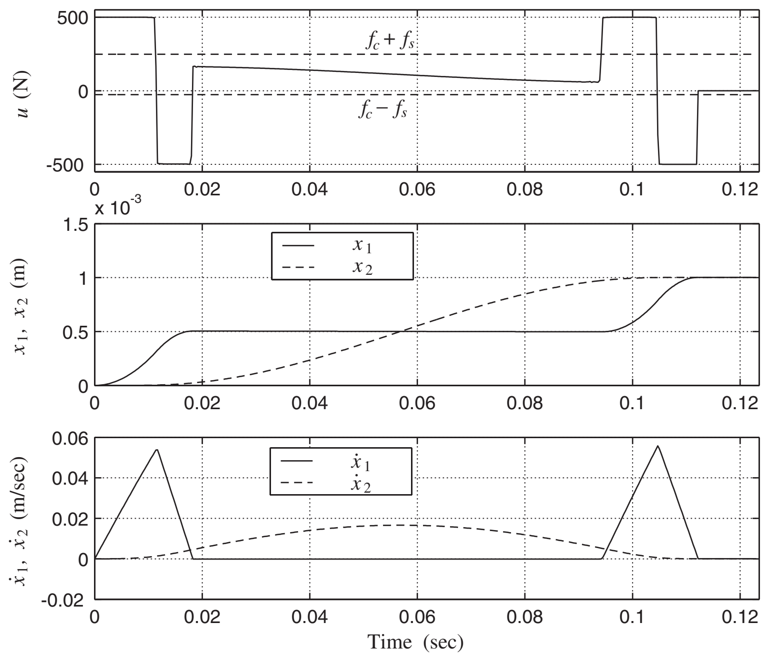

Figure 4. Control input and responses, displacement 0.001 m.

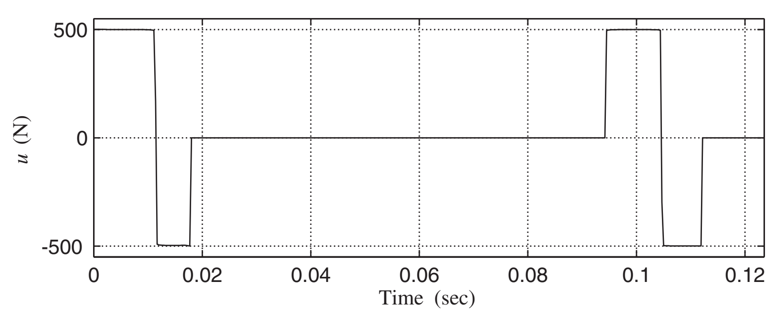

Figure 5. Equivalent input force, $d = 0.001$ m.

where, $k = 1, 2, \ldots, N$. Therefore, the equivalent control profile in Figure 4 becomes bang off bang as shown in Figure 5.

The third simulation is with the initial and final condition of $\mathbf{x}(0) = [0\ 0\ 0\ 0]^{\mathrm{T}}$ and $\mathbf{x}(t_{\mathrm{f}}) = [0.01\ 0.01\ 0\ 0]^{\mathrm{T}}$. The corresponding input history and corresponding response plots are shown in Figure 6. Input profile shown in Figure 6 shows that the compensation force to maintain the stiction of the first mass coincides with the plot of $f_{\mathrm{c}} - k(x_2 - x_1)$. However, the input during stiction is not always within the upper and lower bounds. Since the Coulomb friction disappears during the stiction, only spring force needs to be compensated, for the first mass to stay stuck. Therefore, Coulomb friction is subtracted from the input profile during the

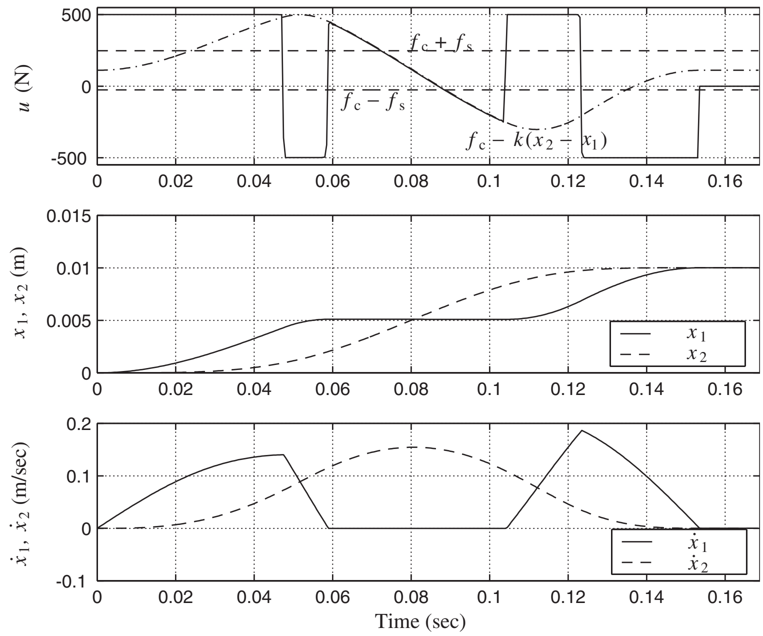

Figure 6. Control input and response, displacement 0.01 m.

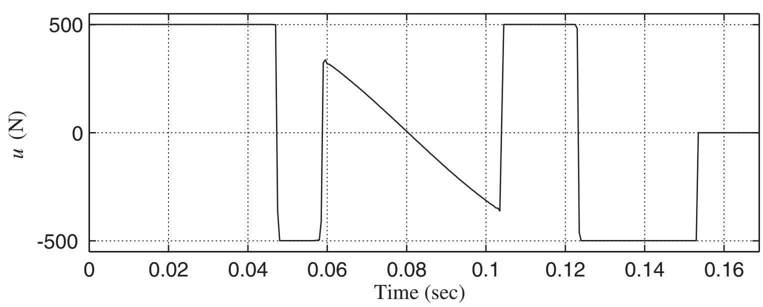

Figure 7. Equivalent input force, $d = 0.01$ m.

stiction and the equivalent input profile is shown in Figure 7. It is also possible to turn off the controller when the input is within the bounds (Equation (15)), however, turning off the controller near the bounds may result in the movement of the first mass when there is a slight variation of the static friction value.

4. POSITIVE PULSE APPROACH

For a floating oscillator problem without friction, negative control input is required to bring the system back to the rest at the final time. However, controller for a floating oscillator with friction does not necessarily require negative force input because friction is acting in the negative direction of the manoeuvre. In addition, applying negative pulse may cause velocity reversals in the small velocity region when there is an error in the model parameters for the controller design and/or there is a variation in Coulomb friction. The new bounds on input for this approach are

$$0 \leqslant u_k \leqslant u^p \quad (k = 1, 2, \ldots, N) \tag{16}$$

With the new input bounds, the control input history and its corresponding response is shown in Figure 8 with initial condition and final condition of $\mathbf{x}(0) = [0\ 0\ 0\ 0]^{\mathrm{T}}$ and $\mathbf{x}(t_{\mathrm{f}}) = [0.001\ 0.001\ 0\ 0]^{\mathrm{T}}$. After the first pulse in the input profile, the control input is turned off to bring the first mass to a stop and then starts to compensate Coulomb friction and spring force. However, the controller does not need to compensate because the control input satisfies Equation (15). Therefore, the equivalent control profile becomes two positive pulses with different pulse widths as shown in Figure 8.

The second simulation for the positive pulse approach is performed with the initial and final states of $\mathbf{x}(0) = [0\ 0\ 0\ 0]^{\mathrm{T}}$ and $\mathbf{x}(t_{\mathrm{f}}) = [0.01\ 0.01\ 0\ 0]^{\mathrm{T}}$. The input force plot in Figure 9 shows that the compensation of spring force is needed to prevent velocity reversals of the first mass and to stay stuck. The responses and the equivalent input history plot are also shown in Figure 9.

The positive pulse approach can be applied for larger displacement; however, there is a large increase in final time, t_{f}, because the velocity gets larger and available friction force is limited. Therefore, the positive pulse solution is more suitable for small displacement or actuating near the reference point. To illustrate this, the variation of the switching time and final time is plotted for different displacements in Figure 10. It is shown that the time between t_4 and t_{f} become

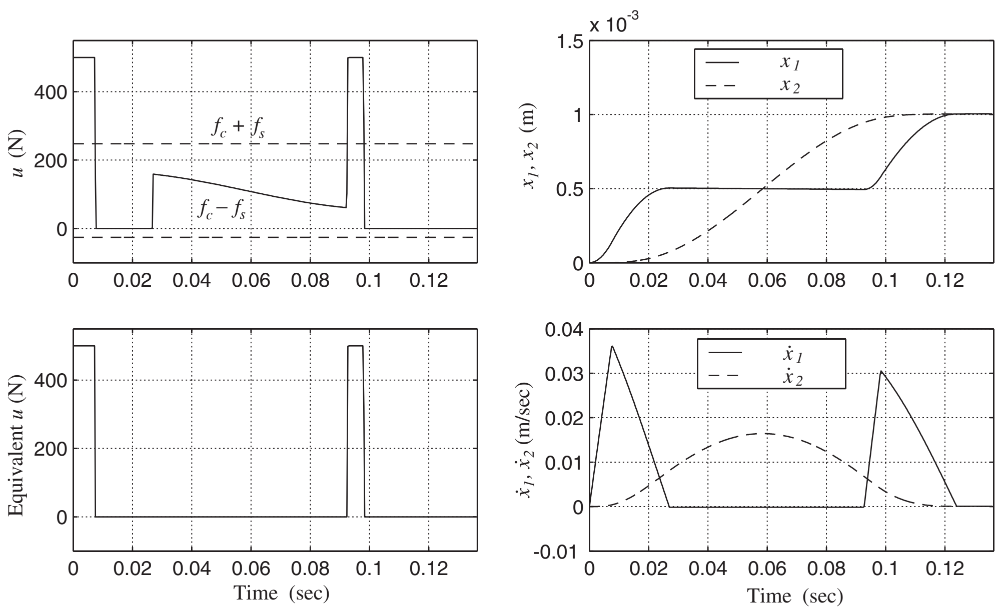

Figure 8. Control input and responses, displacement 0.001 m.

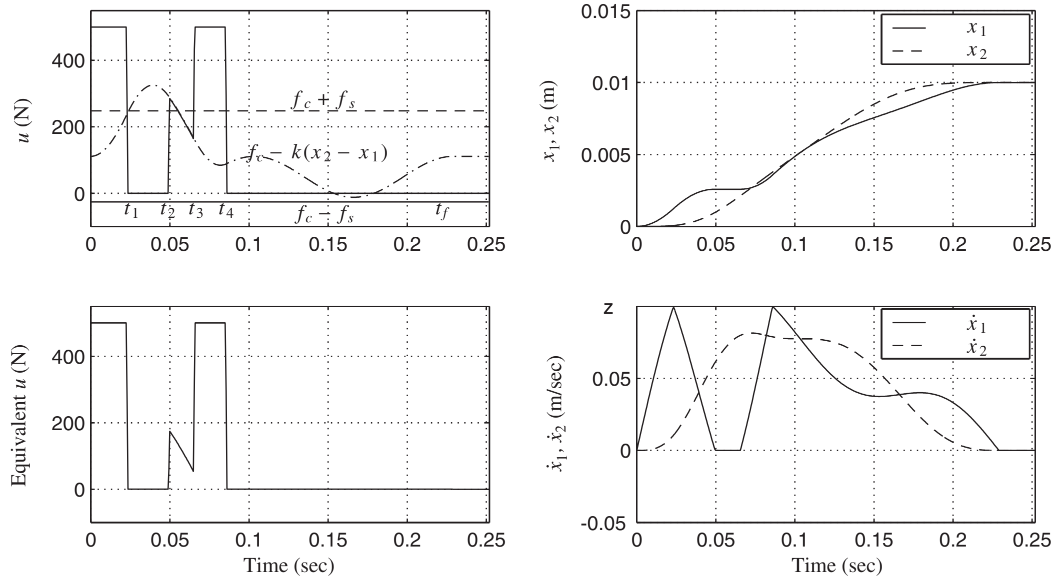

Figure 9. Control input and responses, displacement 0.01 m.

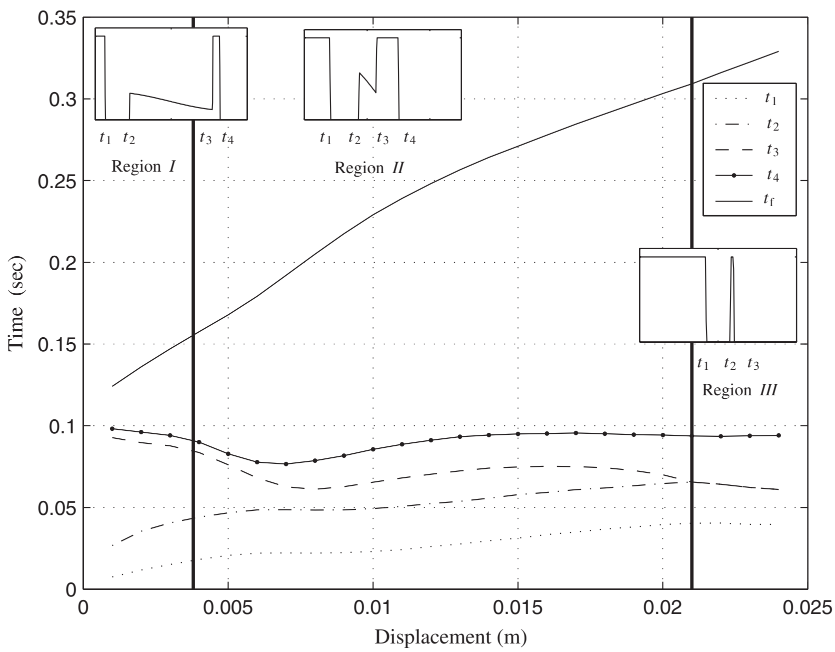

Figure 10. Switching and final time vs. displacement.

larger as displacement becomes bigger. In Figure 10, Region I is where the spring force generated during stiction is not large enough to overcome static friction. In this region, two positive pulses are used. In Region II, the spring force on the first mass is compensated between t_2 and t_3 to stay stuck. In Region III, velocity of the first mass is always positive and the resulting control is a two positive pulse profile. The control profile for each region is also shown in Figure 10 with the switch times.

5. DESENSITIZED SOLUTION

Insensitivity to an uncertain parameter of the model can be obtained by forcing the sensitivity states at the final time to be zeros [11]. The sensitivity of the states to the ith spring constant is defined as:

$$\mathbf{s} = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \frac{\partial x_1}{\partial k_i} & \frac{\partial x_2}{\partial k_i} & \cdots & \frac{\partial x_n}{\partial k_i} \end{bmatrix}^{\mathrm{T}} \tag{17}$$

The direct differentiation of Equation (1) with respect to the ith spring constant leads to differential equation of the sensitivity states.

$$\begin{aligned}
&m_1\ddot{s}_1 + c_1(\dot{s}_1 - \dot{s}_2) + k_1(s_1 - s_2) = 0\\
&m_2\ddot{s}_2 + c_1(-\dot{s}_1 + \dot{s}_2) + c_2(\dot{s}_2 - \dot{s}_3) + k_1(-s_1 + s_2) + k_2(s_2 - s_3) = 0\\
&\qquad\vdots\\
&m_i\ddot{s}_i + c_{i-1}(-\dot{s}_{i-1} + \dot{s}_i) + c_i(\dot{s}_i - \dot{s}_{i+1}) + k_{i-1}(-s_{i-1} + s_i)\\
&+k_i(s_i - s_{i+1}) + (x_i - x_{i+1}) = 0\\
&m_{i+1}\ddot{s}_{i+1} + c_i(-\dot{s}_i + \dot{s}_{i+1}) + c_{i+1}(\dot{s}_{i+1} - \dot{s}_{i+2}) + k_i(-s_i + s_{i+1})\\
&+k_{i+1}(s_{i+1} - s_{i+2}) + (-x_i + x_{i+1}) = 0\\
&\qquad\vdots\\
&m_n\ddot{s}_n + c_n(-\dot{s}_{n-1} + \dot{s}_n) + k_n(-s_{n-1} + s_n) = 0
\end{aligned} \tag{18}$$

The sensitivity states can be augmented with the states $\mathbf{q}$ to form a new system equation of motion. However, there exists a redundant sensitivity state from a fact that there are $n-1$ flexible modes and there are n sensitivity states. The summation of Equation (18) yields

$$m_1\ddot{s}_1 + m_2\ddot{s}_2 + \cdots + m_s\ddot{s}_s + \cdots + m_n\ddot{s}_n = 0 \tag{19}$$

Equation (19) can be solved for $\ddot{s}_1$ as

$$\ddot{s}_1 = -\frac{m_2}{m_1}\ddot{s}_2 - \cdots - \frac{m_s}{m_1}\ddot{s}_s - \frac{m_n}{m_1}\ddot{s}_n \tag{20}$$

According to Equation (20), the sensitivity state s_1 becomes zero when all the rest of sensitivity states $s_2 \ldots s_n$ become zero at the final time. For the previous example of two-mass-spring system, the sensitivity states are defined as

$$s_1 = \frac{\partial x_1}{\partial k} \quad s_2 = \frac{\partial x_2}{\partial k} \tag{21}$$

The differential equation of the sensitivity states becomes

$$\begin{aligned} m_1\ddot{s}_1 + k(s_1 - s_2) + (x_1 - x_2) = 0 \\ m_2\ddot{s}_2 - k(s_1 - s_2) - (x_1 - x_2) = 0 \end{aligned} \tag{22}$$

Since $\ddot{s}_1 = -(m_2/m_1)\,\ddot{s}_2$, $\dot{s}_1 = -(m_2/m_1)\,\dot{s}_2$ and $s_1 = -(m_2/m_1)\,s_2$ at the final time, Equation (22) reduces to

$$\ddot{s}_2 + \left(\frac{k}{m_1} + \frac{k}{m_2}\right)s_2 - \frac{1}{m_2}(x_1 - x_2) = 0 \tag{23}$$

Equation (23) is now augmented with Equation (14) to form a new set of equations:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{s}_2 \end{bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & k & 0 \\ -\dfrac{1}{m_2} & \dfrac{1}{m_2} & \dfrac{k}{m_1} + \dfrac{k}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (u - f_c) \tag{24}$$

A discrete time state–space representation of Equation (24) is used to formulate the linear programming problem. Figure 11 shows the linear programming results of the two-mass-spring example for three different displacements. As in the previous example, the control input can be turned off if the required force to stay stuck is within the stiction bounds, which is shown as a dashed line in the input profile plots. For the command displacement of $d = 0.1$ m, the desensitized control profile becomes a 9 switch bang–bang profile. The corresponding response plots of x_1 and x_2 are also shown in Figure 11. Figure 12 shows the variation of residual energy with respect to the variation of the spring constant for the regular and desensitized control inputs for $d = 0.001$ m. The residual energy is computed from the non-linear simulation results by summing up the potential and kinetic energy. It is shown that the desensitized control input can handle uncertainty of the spring constant value better that the regular solution.

Positive input can also be used for desensitized control problems. Figure 13 shows the desensitized control input and corresponding response plots of the two-mass-spring problem. For $d = 0.001$ and 0.01, three consecutive pulse input can satisfy the boundary conditions when the static friction force, forces the mass to stay stuck. For $d = 0.1$, five positive pulses are required to satisfy the boundary conditions.

To illustrate that the proposed linear programming approach can be used to design controller for multi-mode systems subject to friction, we consider a three-mass-spring system and design the desensitized controller. The parameter values used in the simulation are chosen as $m_1 = 100$ kg, $m_2 = 100$ kg, $m_3 = 100$ kg, $k = 111111.1111$ kg/s^2, $f_s = 137$ N, $f_c = 111$ N, and

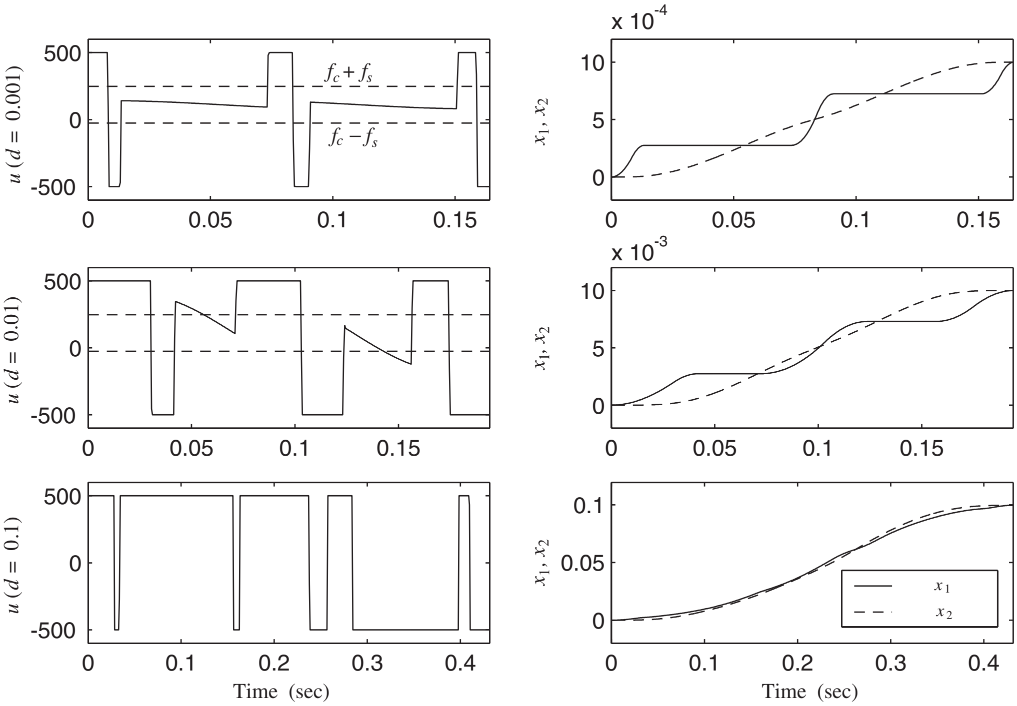

Figure 11. Desensitized input profiles and responses for three displacements.

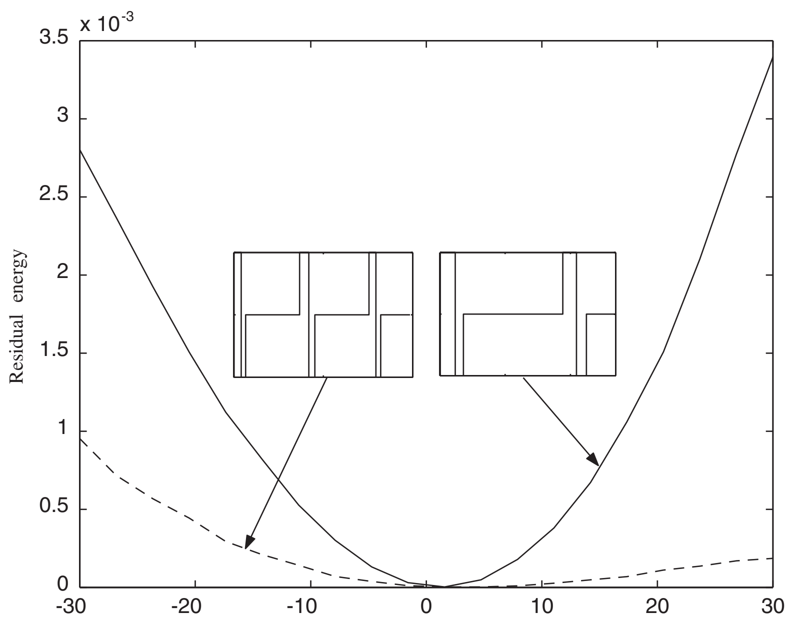

Figure 12. Residual energy vs. spring constant variation.

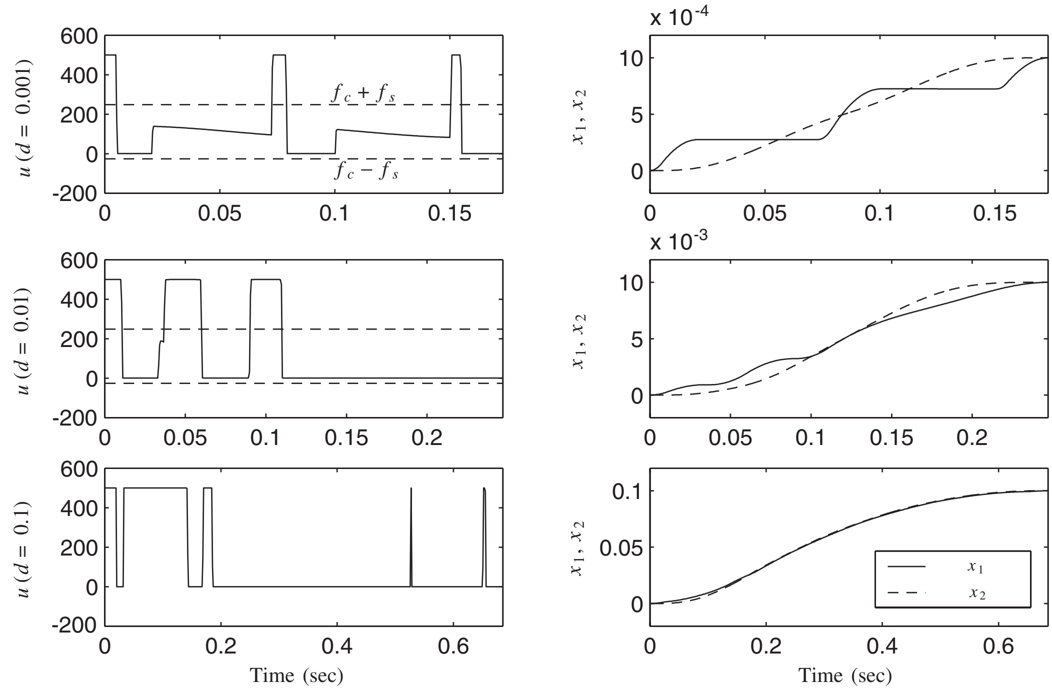

Figure 13. Desensitized positive input profiles and responses for three displacements.

$u^p = 500$ N. In the example problem, the first mass is subject to the control input and friction and the two springs have the same spring constant of k. The augmented equation of motion becomes

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{s}_2 \\ \ddot{s}_3 \end{bmatrix} + \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & k & 0 & 0 \\ -\dfrac{1}{m_2} & \dfrac{2}{m_2} & -\dfrac{1}{m_2} & \dfrac{km_2 + 2km_1}{m_1 m_2} & \dfrac{km_3 - km_1}{m_1 m_2} \\ 0 & -\dfrac{1}{m_3} & \dfrac{1}{m_3} & -\dfrac{k}{m_3} & \dfrac{k}{m_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} (u - f_c) \tag{25}$$

Figure 14 shows the linear programming results for three different manoeuvres. For a small displacement of $d = 0.001$ m, bang-off-bang control profile is obtained. When the displacement gets large ($d = 0.1$ m), the velocity of the frictional mass is always positive and the control profile becomes near time-optimal with a nine switch bang–bang profile.

The positive pulse approach is also applied with the three-mass-spring problem in Figure 15. Five positive pulses are required to finish the manoeuvre of the three masses for displacements

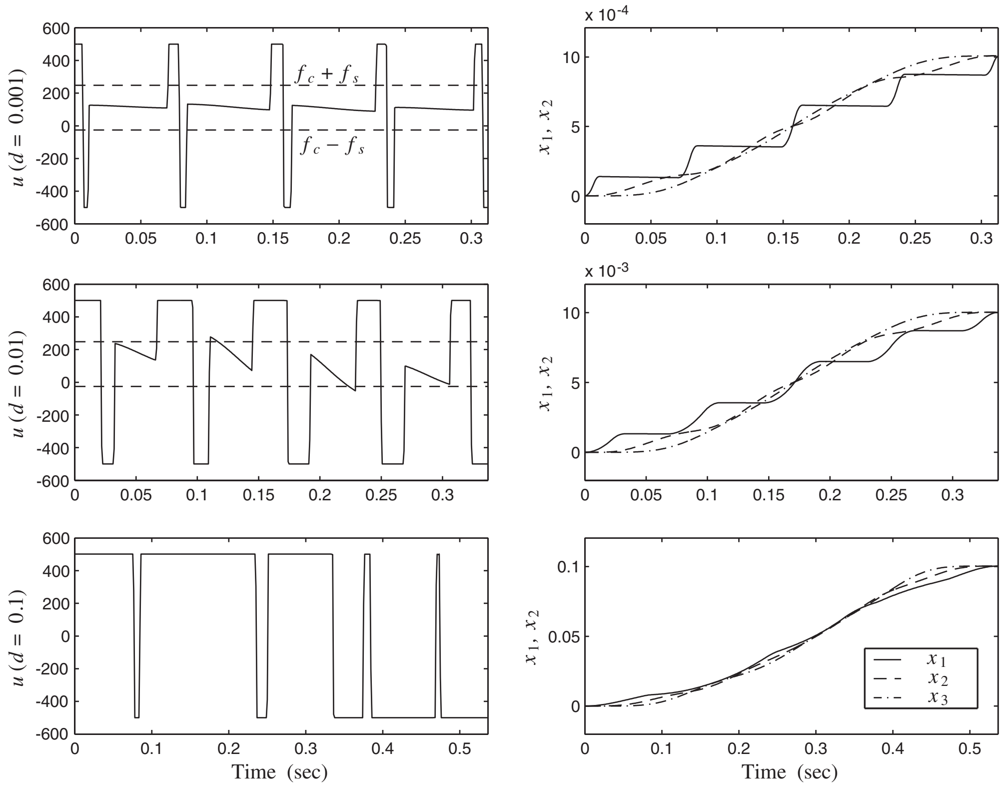

Figure 14. Desensitized positive input profiles and responses for three displacements.

of $d = 0.001$ and 0.01 m. Since the manoeuvre time is large and sampling point is fixed to $N = 400$, an extra pulse is required for $d = 0.1$ m.

To illustrate the insensitivity to the variation of the frequency, three different positive pulse methods are compared in Figure 16 for a two-mass-spring system. The plots on the first row exploit the rigid body assumption of the flexible body used in Reference [2], where the single pulse width t_p is determined by the following equation:

$$t_p = \sqrt{\frac{2d(m_1 + m_2)f_c}{f_p(f_p - f_c)}} \tag{26}$$

The plots on the second and third row show the results from the linear programming problem with $d = 0.001$ m. With the positive pulse approach, two pulses are required to complete the rest-to-rest manoeuvre. The desensitized input requires three consecutive pulses to complete the manoeuvre. For nominal plants which are shown in the second column of the figure, rigid body assumption of the plant causes significant residual vibration at the end of the manoeuvre. With a +20% error in the spring constant, desensitized three pulse control profile performs the best with little residual vibration.

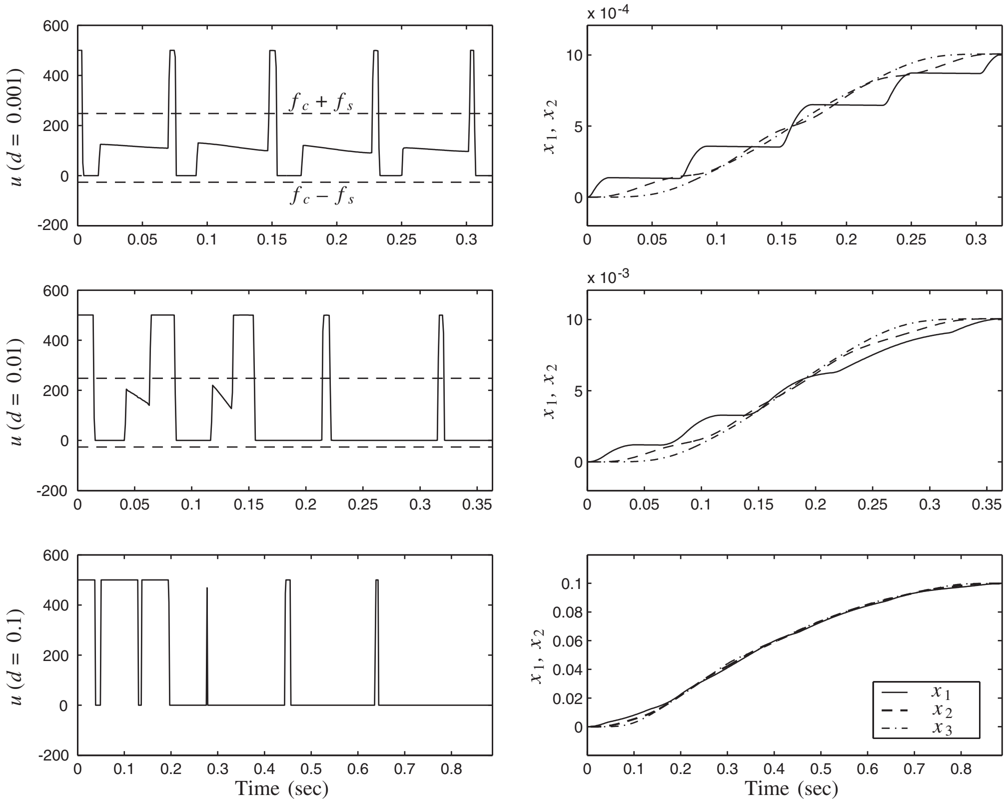

Figure 15. Desensitized positive input profiles and responses for three displacements.

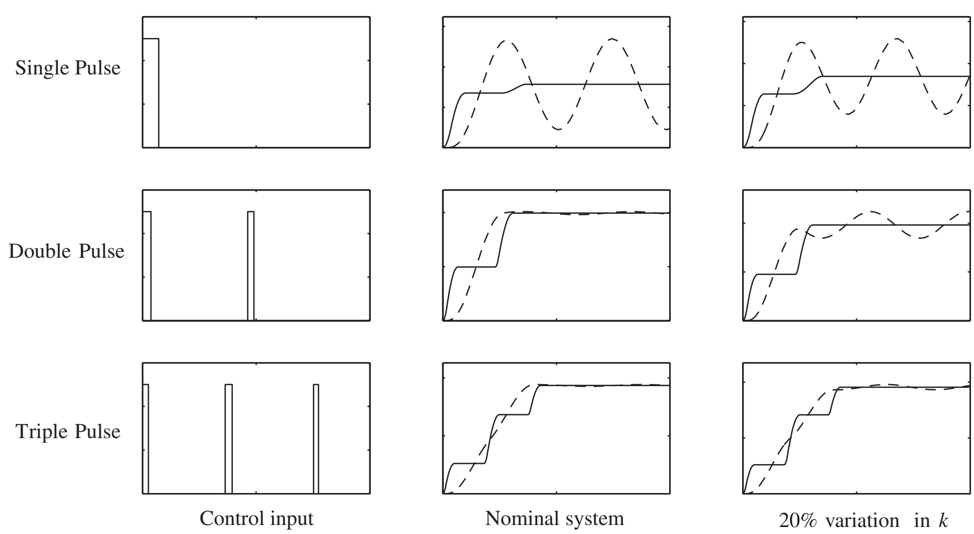

Figure 16. Comparison of pulse width control inputs.

6. CONCLUSION

It is shown in the paper that the linear programming technique is a simple and effective method to find control profiles to manoeuvre flexible structures subject to friction. The non-linear frictional behaviour is linearized by imposing a velocity constraint on the frictional body, which requires that the control and friction are acting on the same body. Solution of the linear programming problem for the frictional system results in a near time optimal control profile, if the velocity constraint of the frictional body is not active. However, it is shown that stiction of the frictional body occurs for certain manoeuvres. If the static friction force is large enough to force the mass to stay stuck, the control input can be turned off, which will result in a bang-off-bang control profile. For the positive pulse approach, the control profile is characterized by consecutive pulses, which can be used for the pulse width control design. Uncertainty in the model parameters can be handled by adding sensitivity states and forcing them to zero at the end of the manoeuvre. It is shown that the residual vibration at the end of the manoeuvre is smaller for desensitized control input in the presence of frequency error of the flexible system.

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