thrust-induced braking acceleration. Therefore, including air drag, the equations of motion now become

$$\dot{v} = -(\alpha + \beta v^2)g + g\cos\psi \tag{4a}$$

$$v\dot{\psi} = -g\sin\psi \tag{4b}$$

where

$$\beta = \frac{\rho_* C_D A}{2mg} \tag{5}$$

The drag coefficient C_D, mass m, and aerodynamic reference area A are all assumed to be constant, as is the local atmospheric density ρ_*. Again, from Eqs. (4) the vehicle flight-path angle may be used as the independent variable to obtain a single first-order equation, viz.,

$$\frac{dv}{d\psi} = f_1(\psi)v + f_2(\psi)v^3 \tag{6}$$

where

$$f_1(\psi) = \alpha\,\mathrm{cosec}\,\psi - \cot\psi \tag{7a}$$

$$f_2(\psi) = \beta\,\mathrm{cosec}\,\psi \tag{7b}$$

Equation (6) is a form of Bernoulli's equation that has a closed-form solution.[7] It can be demonstrated that the nonlinear equation can be transformed into a linear equation with an integrating factor through an appropriate variable transformation. The general solution is then given by

$$v(\psi)^{-2} = \kappa\exp\left(-2\int f_1\,d\psi\right) - 2\exp\left(-2\int f_1 d\psi\right) \times\left[\int \exp\left(2\int f_1\,d\psi\right) f_2\,d\psi\right] \tag{8}$$

Evaluating the integrals and determining the constant of integration κ from the boundary conditions, the solution is found to be

$$v(\psi)^{-2} = \left[\frac{\cos(\psi/2)}{\cos(\psi_0/2)}\right]^{2(1+\alpha)}\left[\frac{\sin(\psi/2)}{\sin(\psi_0/2)}\right]^{2(1-\alpha)} \times\left[v_0^{-2} + \frac{\beta\cos^2(\psi_0/2)}{\alpha-1} + \frac{\beta\sin^2(\psi_0/2)}{\alpha+1} + \frac{2\beta\cos^2(\psi_0/2)\sin^2(\psi_0/2)}{\alpha-\alpha^3}\right] - \frac{\beta\cos^2(\psi/2)}{\alpha-1} - \frac{\beta\sin^2(\psi/2)}{\alpha+1} - \frac{2\beta\cos^2(\psi/2)\sin^2(\psi/2)}{\alpha-\alpha^3} \tag{9}$$

which clearly reduces to the vacuum case defined by Eq. (3) as $\beta \to 0$.

The velocity-flight-path-angle profile is shown in Fig. 2 for a range of drag parameters β. It can be seen that as the effect of air drag increases, the trajectory curves more quickly toward the local vertical. Because the effect of drag is to increase the effective thrust-weight ratio of the vehicle, the descent maneuver may be completed with a lower thrust-induced acceleration than would be required for the vacuum descent case.

Conclusions

It has been demonstrated that the conventional solution for a gravity turn maneuver in vacuum may be extended to include descent to the surface of a body with an atmosphere. With the assumption of quadratic air drag, the resulting equations of motion are shown to be a form of Bernoulli's equation, which has a closed analytical solution. With the vehicle velocity available as a function of flight-path angle, the solution for the altitude and time variables is then reduced, in principle, to a set of quadrature integrations.

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Fuel/Time Optimal Control of Spacecraft Maneuvers

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I. Introduction

CONSIDERABLE interest had developed in the study of optimization theory as applied to the spacecraft system by the early 1960s. The basic theory in determining the extrema of the optimal control problems has been developed for nonsingular[1,2] and singular[3] controls. Computation difficulties have plagued the study of time- and fuel-optimal control problems, particularly for systems with nonlinear dynamics. However, there has been a resurgence of interest in the design of controllers for spacecraft reorientation maneuvers in the past decade.[4–10] Among these studies, the optimization objectives have included the maneuver time,[4,5,7] the fuel consumed,[6,8,9] and the weighted fuel/time cost function.[10] In addition, the singular controls of both time- and fuel-optimal controls have been analyzed for spacecraft reorientations.[9,11]

This Note addresses the problem of designing fuel/time-optimal controllers for spacecraft undergoing rest-to-rest maneuvers. A modified switch time optimization (STO) algorithm[12] is used to solve the problem of reorienting an inertially symmetric spacecraft with weighted fuel/time cost function from an initial state of rest to a final state of rest. In this work, we do not study controls with singular arc and, therefore, assume that the fuel/time-optimal control profile is bang-off-bang.[10,13,14] As the weight on the fuel, α, is increased from zero, it is shown that the number of switches in the control profiles, for an inertially symmetric spacecraft, varies from 5 to 10 to 9. Beyond a specific value of α, it is shown that the eigenaxis control with two switches is the optimum.

II. Problem Formulation

The Euler's rotational equations of motion for an inertially symmetric rigid spacecraft with principal body axes at the center of mass are

$$\dot{\omega} = u \tag{1}$$

where $\omega^T = [\omega_1\ \ \omega_2\ \ \omega_3]$ is the angular velocity vector and $u^T = [u_1\ \ u_2\ \ u_3]$, the control vector, is subject to the constraints

$$-1 \le u_i \le 1 \qquad i = 1, 2, 3 \tag{2}$$

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The associated kinematic equations of motion are

$$\dot{q} = \tfrac{1}{2}\Omega q \tag{3}$$

where

$$\Omega = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{4}$$

and $q^T = [q_0 \ q_1 \ q_2 \ q_3]$ are the quaternions that represent orientation between the Euler axis and the body-fixed reference frame.

The problem of design of weighted fuel/time-optimal controllers for spacecraft reorientation can be stated as follows: Determine the controls that drive the system states (ω, q) described by Eqs. (1) and (3) from their specified initial conditions (ω_0, q_0) to their final conditions (ω_f, q_f) while minimizing the cost function

$$J = \int_0^{t_f} \left(1 + \alpha \sum_{i=1}^{3} |u_i|\right) \mathrm{d}t \tag{5}$$

subject to control constraints [Eq. (2)]. A modification of the STO algorithm that constructs an optimal solution based on the first-order gradient method by integrating the state equations forward in time using initial guesses of the final time and switch times of the controls, and the costates backward in time, is used to determine the optimal control profile. The errors in the terminal constraints are used to update the estimated values of the switch times and the maneuver time for each iteration until convergence. In the fuel/time optimal problem, we assume the control profiles are bang-off-bang, so that we have to investigate only the behavior close to the switch times as the STO algorithm does for the time-optimal control problem.

III. Numerical Examples

The problem addressed in this study is the design of a weighted fuel/time optimal controller for an inertially symmetric spacecraft undergoing a 180-deg rest-to-rest reorientation. The problem is subject to the following boundary conditions.

1) Initial conditions:

$$\omega_1(0) = \omega_2(0) = \omega_3(0) = 0$$

and

$$q_0(0) = 1, \qquad q_1(0) = q_2(0) = q_3(0) = 0$$

2) terminal conditions:

$$\omega_1(t_f) = \omega_2(t_f) = \omega_3(t_f) = 0$$

and

$$q_0(t_f) = q_1(t_f) = q_2(t_f) = 0, \qquad q_3(t_f) = 1$$

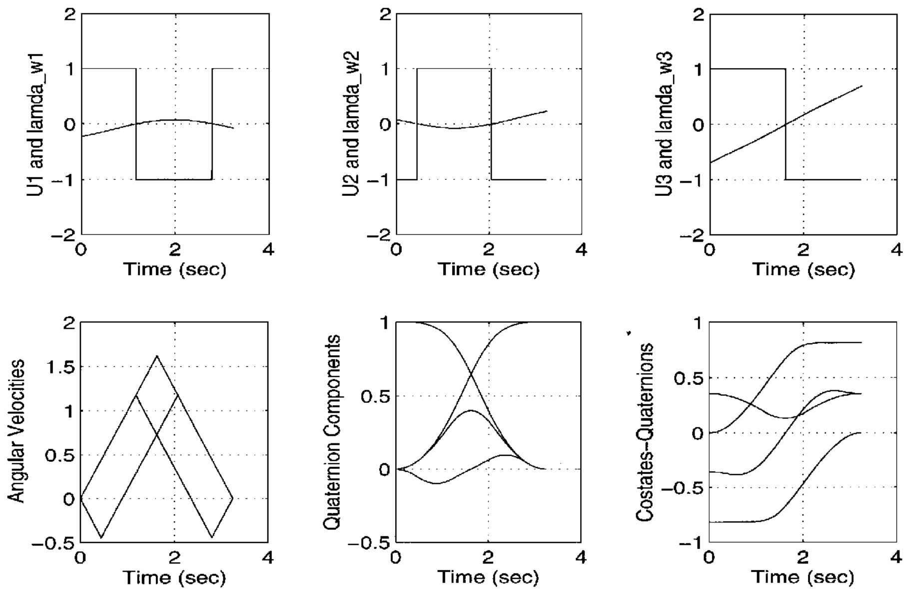

Fig. 1 Controls, states, and costates for a 180-deg, rest-to-rest maneuver with α = 0.

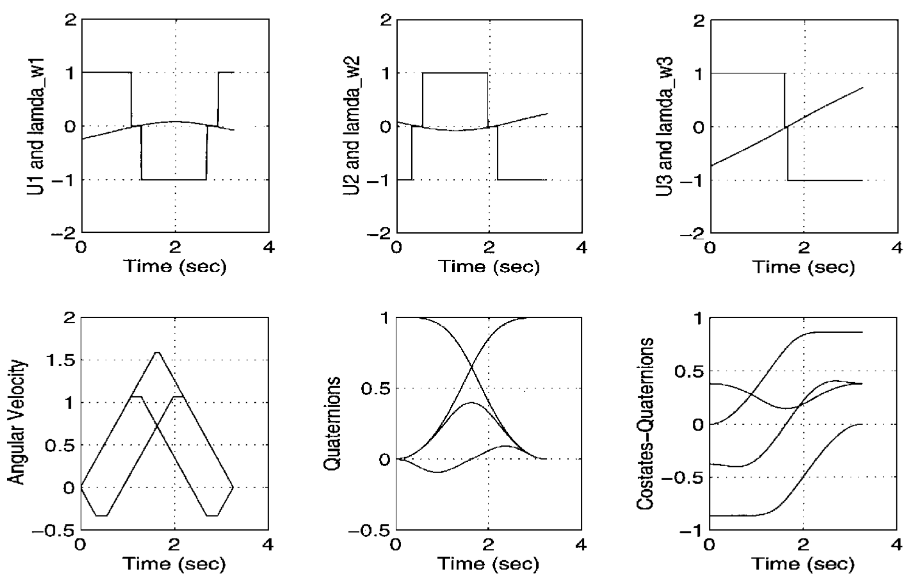

Fig. 2 Controls, states, and costates for a 180-deg rest-to-rest maneuver with α = 0.02.

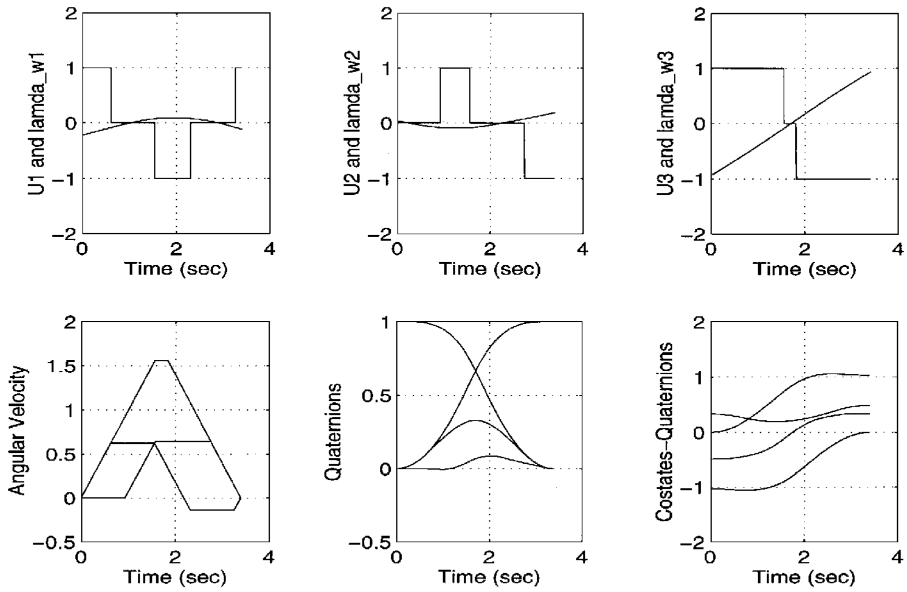

Fig. 3 Controls, states, and costates for a 180-deg rest-to rest maneuver with α = 0.08.

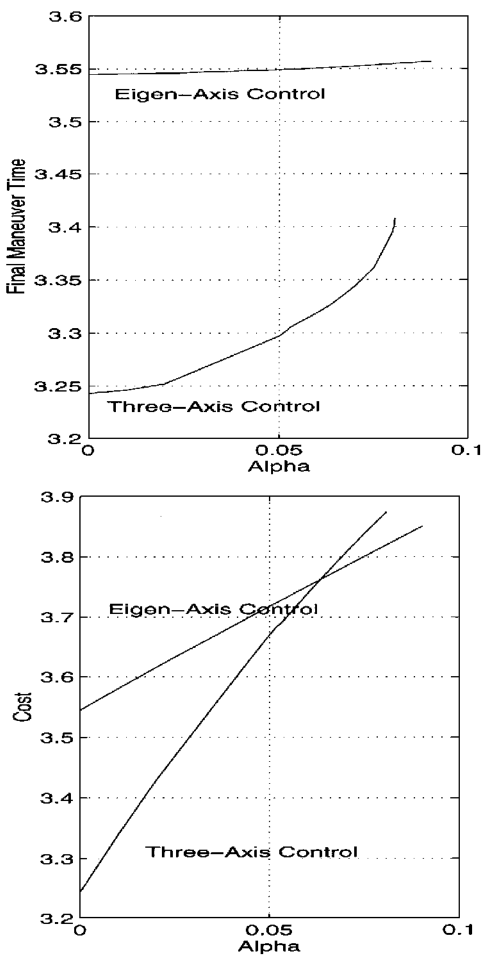

Fig. 4 Comparison using three-axis and eigenaxis controls.

and control constraints:

1) $-1 \leq u_i \leq 1$, $i = 1$, 2, and 3 for three-axis control;

2) $-1 \leq u_3 \leq 1$ for eigenaxis control.

The modified STO algorithm will be used to solve both the time-optimal and fuel/time-optimal control problems. A comparative study of three-axis and eigenaxis control will also be carried out.

The time-optimal control profile is solved and the maneuver time is shown to be $t_f = 3.243$, which is the same as that shown in Refs. 4 and 10. Figure 1 illustrates the time histories of controls, states, and costates. The switching function, $\partial H/\partial u_i$, is the same as the costate of the associated angular velocity in this application.

Because the time-optimal control consists of 5 switches, including fuel into the cost function should lead to a control profile that is characterized by 10 switches.[13] This conjecture is verified for the case of $\alpha < 0.053$, where the control profile maintains 10 switches for the three-axis control. A typical set of time responses for $\alpha = 0.02$ is shown in Fig. 2. When α is in the range 0.053–0.0806, there are only nine switches for the three-axis control. Figure 3 illustrates the nine-switch control with the corresponding states and costates for the case of $\alpha = 0.08$.

Bilimoria and Wie[4] have shown that the eigenaxis control is not the time-optimal control for the prescribed maneuver. Here, we study the variation of the fuel/time cost as a function of α for eigenaxis control. The variation of costs and final maneuver times for different α for three-axis and eigenaxis controls are shown in Fig. 4. Results indicate that the eigenaxis control turns out to be the optimal control for $\alpha \geq 0.063$. The reason for this is that the final maneuver time is increasing more sharply with α for the three-axis control compared to the eigenaxis controller.

IV. Conclusions

A modified STO algorithm to solve the weighted fuel/time optimal control problem has been developed. An inertially symmetric spacecraft undergoing a rest-to-rest reorientation maneuver with three independent bounded impulsive controls is investigated. Solutions to both time-optimal and fuel/time-optimal control problems are analyzed using the modified STO algorithm.

Results from this study illustrate the variation of the switch-time of the control profile from a 5-switch for the time-optimal case to a 10-switch for small α. Increasing α beyond 0.053 results in a transition from the 10-switch profile to a 9-switch profile. For $\alpha > 0.063$, it is shown that the eigenaxis control with two switches is the optimum.

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Analytical Solutions for Exponentially Correlated Acceleration Tracking Filters

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I. Introduction

IN earlier papers, Gupta and Ahn[1] and Gupta[2] have presented an analytical solution of the steady-state exponentially correlated acceleration (ECA) filter with the target position as the only measurement. In radar applications, however, Doppler measurements are also available in addition. Fitzgerald[3] has presented computed steady-state data for a simple form of the ECA target tracking filter, which utilized both position and velocity measurements. In a recent paper, Ramachandra et al.[4] have presented the steady-state solution of a three-state Kalman tracking filter that utilized both position and velocity measurements and a constant-acceleration model.

The objective here is to extend the case of a closed-form steady-state solution for the discrete ECA tracking filter by using the MacFarlane–Potter–Fath eigenstructure method.[5] The ECA tracking filter estimates the target position, velocity, and acceleration in a track-while-scan system and utilizes the target position and velocity measurements as inputs to the tracking filter. When the variance of the velocity measurement error σ_2 goes to infinity, the results are shown to be in agreement with the analytic solution given by Gupta[2] and with the numerical solution given by Fitzgerald.[3]

II. Discrete Exponentially Correlated Acceleration Tracking Filter

The following discrete ECA tracking filter model is considered[6]:

$$x(k+1) = \Phi(T_s)x(k) + v(k) \qquad z(k) = Hx(k) + w(k)$$
$$\Phi(T_s) = \exp(FT_s) \tag{1}$$

with

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau \end{bmatrix} \qquad Q(T_s) = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}$$

$$\Phi(T_s) = \begin{bmatrix} 1 & \tau\theta & \tau^2\phi_1 \\ 0 & 1 & \tau(1-\phi_3) \\ 0 & 0 & \phi_3 \end{bmatrix} \tag{2}$$

$$\Phi^{-1}(T_s) = \begin{bmatrix} 1 & -\tau\theta & -\tau^2\psi_1 \\ 0 & 1 & \tau(1-\psi_3) \\ 0 & 0 & \psi_3 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R(T_s) = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

and

$$\theta = T_s/\tau, \qquad \phi_1 = \theta - 1 + \phi_3, \quad \text{and} \quad \phi_3 = \exp(-\theta)$$
$$\psi_1 = \theta + 1 - \psi_3, \qquad \psi_3 = \exp(\theta), \quad \text{and} \quad \phi_3\psi_3 = 1 \tag{3}$$

Here $\Phi(T_s)$ is the dynamic state transition matrix of the system, H is the measurement matrix, $Q(T_s)$ is the process noise covariance matrix that is symmetric and nonnegative definite and q_{ij} given by Singer,[6] and $R(T_s)$ is the measurement covariance matrix and uncorrelated with $v(k)$.

It is well known that the steady-state Kalman filter for Eqs. (1–3) becomes

$$\hat{x}(k/k) = \hat{x}(k/k-1) + K[z(k) - H\hat{x}(k)] \tag{4}$$

where the Kalman gain matrix K is given by

$$K = PH^TR^{-1} = \begin{bmatrix} p_{11}/R_1 & p_{12}/R_2 \\ p_{21}/R_1 & p_{22}/R_2 \\ p_{31}/R_1 & p_{32}/R_2 \end{bmatrix} \tag{5}$$

where P is the a posteriori covariance matrix of the estimation errors.

The five parameters used to describe this problem[3] are rms target acceleration σ_a, correlation time of target acceleration τ, sampling time T_s, rms position measurement error σ_{mp}, and rms velocity measurement error σ_{mv}.

In Eq. (3), $R_1 = \sigma_{mp}^2$, $R_2 = \sigma_{mv}^2$, and we define the three dimensionless parameters as

$$p_1 \equiv \tau/T_s \tag{6}$$

$$p_2 \equiv \frac{T_s^2\sigma_a}{\sigma_{mp}} \tag{7}$$

$$p_3 \equiv \frac{T_s\sigma_{mv}}{\sigma_{mp}} \tag{8}$$

We restrict p_1 (Ref. 3) to a few simple multiples of the critical value $p_{1c} = \tau_c/T_s$. The critical value maximizes the position and velocity errors of the filter. Values determined empirically are well approximated by the equation[3]

$$p_{1c} = \tau_c/T_s = \left[0.56 + 3.4p_2^{-0.86}\right]^{\frac{1}{2}} \tag{9}$$

III. MacFarlane–Potter–Fath Eigenstructure Method

The steady-state solution of the time-invariant matrix Riccati equation was discovered independently by MacFarlane et al.[5] The solution $P(\infty)$ of the steady-state matrix Riccati equation in discrete time is formalized as Lemma 1.

Lemma 1 (Ref. 5). If W_{11} and W_{21} are $n \times n$ matrices such that W_{21} is nonsingular and

$$H_f\begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} = \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix}D \tag{10}$$

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