Time-Optimal Output Transition for Minimum Phase Systems

Jennifer Haggerty
Graduate Research Assistant
Department of Mechanical and
Aerospace Engineering
University at Buffalo
Buffalo, NY 14260
Email: jrh6@buffalo.edu

Tarunraj Singh *

Professor, Fellow of ASME
Department of Mechanical and
Aerospace Engineering
University at Buffalo
Buffalo, NY 14260
Email: tsingh@buffalo.edu

ABSTRACT

The time-optimal output transition control problem for stable or marginally stable systems with minimum-phase zeros is discussed in this paper. A double integrator system with a real left-half plane zero is used to illustrate the development of the time-optimal output transition controller. It is shown that an exponentially decaying post-actuation control profile is necessary to maintain the output at the desired final location. It is shown that the resulting solution to the output transition time-optimal control profile can be generated by a time-delay filter whose zeros and poles cancels the poles and zeros of the system to be controlled. The design of the time-optimal output transition problem is generalized and illustrated on the benchmark floating oscillator problem.

1 Introduction

Feedforward control is now a reliable approach for precision motion control in applications ranging from hard disk drives [1], wafer scanners [2], cranes [3], to flexible space structures [4]. This includes shaping the reference input for a stable system, i.e., feedback stabilized system or an open loop-control such as time-optimal control which provides the nominal trajectories which are followed by a perturbation feedback controller. The input-shaping/time-delay filtering [5, 6] approach which eliminates residual vibration by a simple phase shifted harmonic cancelation has been extensively studied. Issues dealing with desensitizing the reference shaper to model parameter uncertainties has been the focus to permit robust design [7, 8] while techniques to minimize excitation of un-modeled modes has been addressed by limiting the jerk [9] or by using smooth trajectories [10]. Modifications to the traditional input shaper design include work by Dijkstra and Bosgra [11] where they illustrate the use of iterative learning control to design an input shaper to eliminate residual vibrations in rest-to-rest maneuvers of wafer scanners. An adaptive input shaping technique was presented by Bodson [12] where the traditional time-delay filter and a second order pre-filter are designed to eliminate residual vibrations.

Another class of open-loop controller which focuses on rest-to-rest maneuvers such a time-optimal [13–15] fuel-time optimal [16, 17], jerk-limited time-optimal [18, 19] which explicitly includes control constraints has also been the focus of numerous researchers over the past three decades. All the aforementioned papers deal with state-to-state transition. Often one is interested in regulating specific outputs which can be a function of multiple states. For systems whose transfer

*Corresponding author

functions are characterized by zeros, the output is a function of multiple states. For such systems, one can pose an optimal control problem which endeavors to transition the output from a state of rest to a state of rest. This optimal output transition problem has been addressed by Iamratanakul et al. [20] for a dual-stage disk drive where pre- and post-actuation is used to minimize the energy consumed in the output-to-output transition problem. Iamratanakul and Devasia [21] extended the minimum-energy control profile to a weighted time/energy cost function. They show that using pre- and post-actuation results in reducing the weighted time/energy cost function for output-to-output transition relative to state-to-state transition. Devasia [22] developed a time-optimal design strategy for system with rigid and flexible modes and illustrated the technique on a benchmark floating oscillator problem with multiple inputs. The output-to-output transition problem was shown to be a bang-bang profile in the transition time window and an output-maintaining inverse input law was derived to synthesize the pre- and post-actuation control profiles.

This article focuses on systems with minimum-phase zeros, i.e., zeros in the left half of the complex plane. To help illustrate the motivation for the parameterizations of the time-optimal output-to-output transition control profile, a double integrator with a left half plane zero is considered. The optimal control is shown to be identical to one synthesized from a time-delay filter which is designed to cancel the poles and zeros of the system via the zeros and poles of the transfer function of the time-delay filter. The change in the structure of the post-actuation time-optimal control profile as a function of the transition maneuver is illustrated for the double integrator problem. A generalization of the design approach is presented followed by the illustration of the design of a post-actuation time-optimal controller for the benchmark floating oscillator problem.

2 Problem Formulation

This section will focus on the development of a post-actuation time-optimal controller for a double integrator with a left half plane zero. Closed form solutions to the output transition minimum-time control will be derived. Variation in the structure of the time-optimal control as a function of the maneuver distance will be illustrated. The design of the post-actuation control is initiated in the time-domain since it permits a clear motivation for the form of the post-actuation control. It also help transition and illustrate the parity in formulating the post-actuation control in the frequency domain, leading to the final generalization in the frequency domain.

2.1 Time Domain Development

Consider the time optimal control of the system:

$$\ddot{y}(t) = k\dot{u}(t) + \alpha u(t) \tag{1}$$

to transition from a initial state of rest to a terminal state where the output is at rest. Assume k and α are greater than zero, resulting in a minimum phase zero of the transfer function relating the input u to the output y. Rewriting the system in state space form, the time-optimal control problem can be stated as:

$$\min \; J = t_f \tag{2a}$$

subject to

$$\begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{2b}$$

$$y = \begin{bmatrix} \alpha & k \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \tag{2c}$$

$$\phi_1(0) = \phi_2(0) = 0, \text{ and } y(t_f) = y_f, \dot{y}(t_f) = 0, \;\; \forall t \geq t_f \tag{2d}$$

$$-1 \leq u(t) \leq 1 \;\; \forall t \tag{2e}$$

Since the terminal constraint is:

$$\dot{y}(t_f) = \alpha\phi_2(t_f) + ku(t_f) = 0, \tag{3}$$

and with the knowledge that $u(t_f)$ is constrained to lie between -1 and 1, and assuming k is positive, Equation (3) can be rewritten as the inequality constraints:

$$\alpha\phi_2(t_f) - k \le \dot{y}(t_f) = 0 \le \alpha\phi_2(t_f) + k \tag{4}$$

using the limiting value of u. The resulting inequality constraints are:

$$\alpha\phi_2(t_f) - k - \dot{y}(t_f) \le 0 \tag{5}$$
$$-\alpha\phi_2(t_f) - k + \dot{y}(t_f) \le 0 \tag{6}$$

where the terminal velocity of zero has been represented symbolically to permit the determination of an analytical expression of the Lagrange multiplier associated with the inequality constraint. The resulting optimal control problem is:

$$\min \; J = t_f \tag{7a}$$

subject to

$$\begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \tag{7b}$$

$$y = \begin{bmatrix} \alpha & k \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \tag{7c}$$

$$\phi_1(0) = \phi_2(0) = 0, \tag{7d}$$

$$\alpha\phi_1(t_f) + k\phi_2(t_f) = y_f, \tag{7e}$$

$$\alpha\phi_2(t_f) - k - \dot{y}(t_f) \le 0 \tag{7f}$$

$$-\alpha\phi_2(t_f) - k + \dot{y}(t_f) \le 0 \tag{7g}$$

$$-1 \le u(t) \le 1 \;\; \forall t \tag{7h}$$

Including the terminal equality and inequality constraints into the cost function, the augmented cost function can be written as:

$$\begin{aligned} J_a = \; & \nu_1 \left(\alpha\phi_2(t_f) - k - \dot{y}(t_f)\right) + \nu_2 \left(-\alpha\phi_2(t_f) - k + \dot{y}(t_f)\right) + \\ & \beta \left(\alpha\phi_1(t_f) + k\phi_2(t_f) - y_f\right) + \int_0^{t_f} \left(1 + \lambda^T \left(Ax + Bu - \dot{x}\right)\right) dt \end{aligned} \tag{8}$$

where the Lagrange multipliers associated with the terminal inequality constraints: ν_1 and $\nu_2 \ge 0$ and the Hamiltonian is defined as:

$$H = 1 + \lambda^T \left(Ax + Bu\right) \tag{9}$$

The necessary conditions for optimality are:

$$\begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{Bmatrix} = \frac{\partial H}{\partial \lambda} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{10}$$

$$\begin{Bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{Bmatrix} = -\frac{\partial H}{\partial x} = -\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} \tag{11}$$

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} (t_f) = \nu_1 \begin{Bmatrix} 0 \\ \alpha \end{Bmatrix} - \nu_2 \begin{Bmatrix} 0 \\ \alpha \end{Bmatrix} + \beta \begin{Bmatrix} \alpha \\ k \end{Bmatrix} \tag{12}$$

$$u = -sign\left(B^T \lambda\right) \tag{13}$$

$$\nu_1, \nu_2 \ge 0 \tag{14}$$

$$H(0) = 0 \tag{15}$$

Solving the costate equations, we have:

$$\lambda_1(t) = C \tag{16}$$
$$\lambda_2(t) = -Ct + D \tag{17}$$

Since $\lambda_1(t_f) = \beta\alpha$ from Equation (12), we have $C = \alpha\beta$. Exploiting the constraint given by Equation (15), we have:

$$H(0) = 1 + \lambda_2(0) = 1 + D = 0 \tag{18}$$

assuming that the initial control magnitude is 1, resulting in $D = -1$. The resulting time-optimal control is:

$$u = -sign\left(-\alpha\beta t - 1\right). \tag{19}$$

It is clear from the structure of the switching function $-\alpha\beta t - 1$, that the control can switch at most once. Assuming a single switch parameterization of the time optimal control profile as:

$$u(t) = 1 - 2\mathcal{H}(t - T_s) \;\; \forall t \geq 0, \tag{20}$$

where $\mathcal{H}(t - T_s)$ is the Heaviside step function and the control switches at time T_s, we have:

$$-\alpha\beta T_s - 1 = 0 \Rightarrow T_s = -\frac{1}{\alpha\beta}. \tag{21}$$

Integrating the state equations given by Equation (10), we have:

$$\phi_1(t) = \frac{1}{2}t^2 - (t - T_s)^2 \mathcal{H}(t - T_s) \;\; \forall t \geq 0, \tag{22}$$
$$\phi_2(t) = t - 2(t - T_s)\mathcal{H}(t - T_s) \;\; \forall t \geq 0. \tag{23}$$

Since at time t_f, the terminal control is -1, the active terminal constraint is:

$$\alpha\phi_2(t_f) - k - \dot{y}(t_f) = 0 \Rightarrow \alpha(-t_f + 2T_s) - k - \dot{y}(t_f) = 0 \tag{24}$$

which results in the solution:

$$t_f = \frac{2\alpha T_s - k - \dot{y}(t_f)}{\alpha} \tag{25}$$

From the terminal equality constraint:

$$\alpha\phi_1(t_f) + k\phi_2(t_f) = y_f \tag{26}$$
$$\Rightarrow \alpha\left(\frac{1}{2}t_f^2 - (t_f - T_s)^2\right) + k(t_f - 2(t_f - T_s)) = y_f \tag{27}$$

the switch time can be solved in closed form:

$$T_s = \frac{1}{2}\frac{\sqrt{-2k^2 + 4y_f\alpha}}{\alpha}, -\frac{1}{2}\frac{\sqrt{-2k^2 + 4y_f\alpha}}{\alpha}. \tag{28}$$

Equation (21) leads to the closed form solution for the Lagrange multiplier:

$$\beta = -\frac{2}{\sqrt{-2k^2 + 4\alpha y_f}} \tag{29}$$

Since $\lambda_2(t_f)$ from Equation (12) is:

$$\lambda_2(t_f) = (\nu_1 - \nu_2)\,\alpha + \beta k = -\alpha\beta t_f - 1, \tag{30}$$

and with the knowledge that $\nu_2 = 0$, since the corresponding constraint is inactive, we have:

$$\nu_1 = \frac{1}{\alpha} \tag{31}$$

which is a positive number since $\alpha > 0$.

With the knowledge that the Lagrange multipliers are the sensitivity of the cost function to variation in the level of the constraint, we have:

$$\frac{dJ_a}{dy_f} = -\beta \tag{32}$$

$$\frac{dJ_a}{d\dot{y}_f} = -\nu_1 \text{ if constraint u=-1 is active} \tag{33}$$

$$\frac{dJ_a}{d\dot{y}_f} = \nu_2 \text{ if constraint u=1 is active.} \tag{34}$$

Since the Lagrange multiplier β is defined as the sensitivity of the cost function to a perturbation in the constraint level y_f, we have:

$$\beta = -\frac{dt_f}{dy_f} = -\frac{2}{\sqrt{-2k^2 + 4\alpha y_f}}, \tag{35}$$

which matches the solution given by Equation (29).

ν_1 can also be calculated from the sensitivity equation:

$$\nu_1 = -\frac{dt_f}{d\dot{y}_f} = \frac{1}{\alpha}. \tag{36}$$

which matches the solution given by Equation (31). Figure 1 illustrates the variation of the switch time (solid line) and the maneuver time (dashed line) as a function of the final displacement for $\alpha = 1$ and $k = 2$. It should be noted that the switch time and the maneuver time are coincident for a displacement of $y_f = 6$. The structure of the time-optimal post-actuation control for maneuvers smaller than $y_f = 6$ will be presented later.

2.2 Post Actuation

Figure 2 illustrates the terminal equality constraint for the output. It is clear that the terminal states $\phi_1(t_f)$ and $\phi_2(t_f)$ have to lie on the line given by Equation (3) shown by the dashed line. Once the states have reached the constraint line, the states have to evolve such that they slide along the constraint line ensuring that $y(t) = y_f$ and $\dot{y}(t) = 0$ for all time greater than t_f.

Since,

$$\alpha\phi_1(t) + k\phi_2(t) = y_f, \forall t > t_f \tag{37}$$

$$\Rightarrow \phi_2(t) = \frac{y_f - \alpha\phi_1(t)}{k} \tag{38}$$

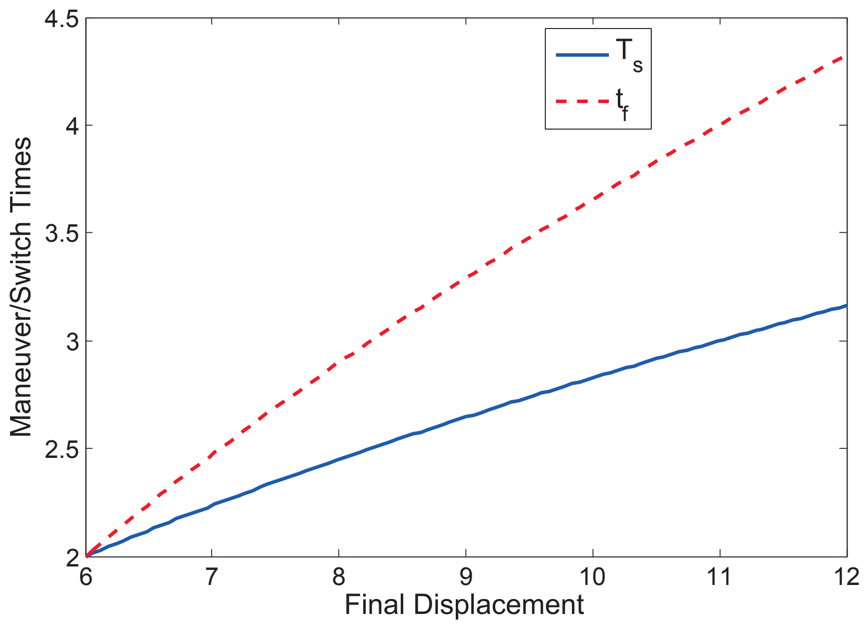

Fig. 1.   Variation of Maneuver and Switch Time

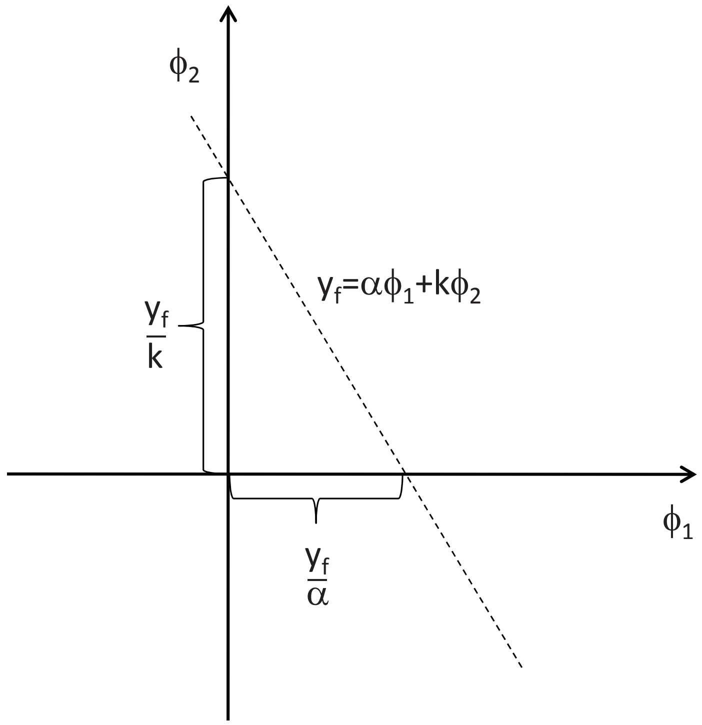

Fig. 2.   Output Constraint for Post-Actuation

and

$$\dot{y}(t_f) = \alpha\phi_2(t) + ku(t) = 0, \forall t > t_f \tag{39}$$

$$\Rightarrow u(t) = -\frac{\alpha}{k}\phi_2(t). \tag{40}$$

Substituting the control given by Equation (40) into the state equation, the resulting evolution of the states after time t_f are:

$$\dot{\phi}_2 = u = -\frac{\alpha}{k}\phi_2 \tag{41}$$

$$\Rightarrow \phi_2(t) = e^{-\frac{\alpha}{k}(t-t_f)}\phi_2(t_f) \;\; \forall t \geq t_f, \tag{42}$$

and

$$\phi_1(t) = -\frac{k}{\alpha}e^{-\frac{\alpha}{k}(t-t_f)}\phi_2(t_f) + \frac{k}{\alpha}\phi_2(t_f) + \phi_1(t_f). \tag{43}$$

The control can now be represented as:

$$u = \dot{\phi}_2 = -\frac{\alpha}{k}e^{-\frac{\alpha}{k}(t-t_f)}\phi_2(t_f). \tag{44}$$

With the knowledge that $u(t_f) = -1$, we have:

$$u(t_f) = -1 = -\frac{\alpha}{k}\phi_2(t_f) \Rightarrow \phi_2(t_f) = \frac{k}{\alpha} \tag{45}$$

or

$$u = -e^{-\frac{\alpha}{k}(t-t_f)} \;\; \forall t \geq t_f \tag{46}$$

is the post-actuation control which constrains the output $y(t)$ to equal y_f for all time greater than t_f. Note that the control given in Equation 46 starts at -1 and transitions exponentially to zero thus satisfying the control constraint given by Equation 7h.

2.3 Frequency Domain Development

Singh and Vadali [15] presented a frequency domain approach for the design of time-optimal controllers. A time-delay filter was parameterized in terms of the switch times and the maneuver time with the knowledge that the time-optimal control profile is bang-bang. The output of the time-delay filter when subject to a step input results in a bang-bang control profile. A parameter optimization problem is posed so as to require zeros of the transfer function of the time-delay filter cancel the poles of the system with an additional constraint to satisfy the terminal maneuver constraint. The same approach of designing a time-delay filter will be used for optimal output-to-output transition control.

With the knowledge that the post-actuation time-optimal control for the second order system given by Equation (1) is:

$$u(t) = 1 - 2\mathcal{H}(t-T_s) + (1 - e^{-\frac{\alpha}{k}(t-t_f)})\mathcal{H}(t-t_f), \tag{47}$$

the frequency domain representation of the control profile is:

$$U(s) = \frac{1}{s}\underbrace{\left(1 - 2\exp(-sT_s) + \frac{\frac{\alpha}{k}}{s+\frac{\alpha}{k}}\exp(-st_f)\right)}_{G_c(s)}. \tag{48}$$

where $G_c(s)$ is the transfer function of a time-delay filter which generates the time-optimal post-actuation control profile.

Note that the transfer function of the time-delay filter includes a pole located at $s = -\frac{\alpha}{k}$ which corresponds to the zero of the system transfer function, in essence canceling the zero of the transfer function with a pole of the time-delay filter. Evaluating the transfer function of the time-delay filter at $s = 0$, we have:

$$G_c(s=0) = 1 - 2e^{-sT_s} + \frac{\frac{\alpha}{k}e^{-st_f}}{(s+\frac{\alpha}{k})} = 1 - 2 + 1 = 0 \tag{49}$$

which cancels one pole of the system transfer function located at the origin. To cancel a second pole at the origin, we require:

$$\frac{dG_c(s)}{ds}(s=0) = 2T_s e^{-sT_s} - \frac{\frac{\alpha}{k}t_f e^{-st_f}}{(s+\frac{\alpha}{k})} - \frac{\frac{\alpha}{k}e^{-st_f}}{(s+\frac{\alpha}{k})^2} \tag{50}$$

$$\Rightarrow = 2T_s - t_f - \frac{1}{(\frac{\alpha}{k})} = 0 \tag{51}$$

when substituting the closed form solution for T_s and t_f given by Equations (25) and (28). Thus the time-optimal post-actuation control can be derived by designing a time-delay filter which uses its poles and zeros to cancel the zeros and poles of the system respectively. Note that the structure of the time-delay filter for output-to-output transition depends on the zeros of transfer function of the system as opposed to the structure of the time-delay filter for state-to-state transition which is a sum of delayed step inputs, which results in a bang-bang control profile.

2.4 Small Maneuvers

Equation 25 presents the relationship between the maneuver time and the switch time:

$$t_f = 2T_s - \frac{k}{\alpha} \tag{52}$$

when $\dot{y}(t_f) = 0$. It can be seen that the maneuver time and the switch time coincide when:

$$T_s = \frac{k}{\alpha} \tag{53}$$

which corresponds to a maneuver of

$$y_f = \frac{3k^2}{2\alpha}. \tag{54}$$

For maneuvers smaller than $y_f = \frac{3k^2}{2\alpha}$, the inequality constraints:

$$\alpha\phi_2(t_f) - k \leq 0 \tag{55}$$

$$-\alpha\phi_2(t_f) - k \leq 0 \tag{56}$$

are not active and consequently, the corresponding Lagrange multipliers ν_1 and ν_2 are zero. This implies that the magnitude of the control at the final time is neither -1 or 1. Assuming the initial control is $u = 1$, the state evolution is given by the equations:

$$\phi_1(t) = \frac{1}{2}t^2 \tag{57}$$

$$\phi_2(t) = t \tag{58}$$

which results in the terminal constraint:

$$\alpha\frac{1}{2}t_f^2 + kt_f = y_f \tag{59}$$

which results in the solution:

$$t_f = \frac{-k \pm \sqrt{k^2 + 2\alpha y_f}}{\alpha}. \tag{60}$$

The final time has to be:

$$t_f = \frac{-k + \sqrt{k^2 + 2\alpha y_f}}{\alpha} \tag{61}$$

since the other solution results in a negative time. Since the output velocity at the terminal time should be 0, we have:

$$\alpha t_f + k u(t_f) = 0, \rightarrow u(t_f) = -\frac{\alpha t_f}{k} = \frac{k - \sqrt{k^2 + 2\alpha y_f}}{k}. \tag{62}$$

The transversality conditions are:

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} (t_f) = \nu_1 \begin{Bmatrix} 0 \\ \alpha \end{Bmatrix} - \nu_2 \begin{Bmatrix} 0 \\ \alpha \end{Bmatrix} + \beta \begin{Bmatrix} \alpha \\ k \end{Bmatrix} \tag{63}$$

and since:

$$\lambda_1(t) = C \tag{64}$$

$$\lambda_2(t) = -Ct + D, \tag{65}$$

$$H(0) = 1 + \lambda_2(0) = 0 \tag{66}$$

we have:

$$C = \alpha\beta \tag{67}$$

$$D = -1 \tag{68}$$

$$\beta = -\frac{1}{k + \alpha t_f} = -\frac{1}{\sqrt{k^2 + 2\alpha y_f}} \tag{69}$$

$$\nu_1 = \nu_2 = 0. \tag{70}$$

Since the Lagrange multiplier β is defined as the sensitivity of the cost function to a perturbation in the constraint, we have:

$$\beta = -\frac{dt_f}{dy_f} = -\frac{1}{2\alpha} \frac{2\alpha}{\sqrt{k^2 + 2\alpha y_f}}, \tag{71}$$

which matches the solution given by Equation (69).

The post-actuation control to maintain $y(t)$ at y_f was shown in Equation (44) to be:

$$u = \dot{\phi}_2 = -\frac{\alpha}{k} e^{-\frac{\alpha}{k}(t - t_f)} \phi_2(t_f), \tag{72}$$

and with the knowledge that:

$$\phi_2(t_f) = t_f = \frac{-k + \sqrt{k^2 + 2\alpha y_f}}{\alpha}, \tag{73}$$

the post-actuation control is:

$$u(t) = \frac{k - \sqrt{k^2 + 2\alpha y_f}}{k} e^{-\frac{\alpha}{k}(t - t_f)}. \tag{74}$$

Since this control is valid for maneuvers less than $y_f \leq \frac{3k^2}{2\alpha}$, it is clear that the coefficient of the exponentially decaying term in the control is less than -1, satisfying the control constraints.

The transfer function of a time-delay filter to generate the time-optimal post-actuation control profile can be shown to be:

$$G_c(s) = 1 - \frac{s\sqrt{k^2 + 2\alpha y_f} + \alpha}{ks + \alpha} exp(-st_f). \tag{75}$$

Evaluating $G_c(s=0)$, and $\frac{dG_c}{ds}(s=0)$, we have:

$$G_c(s=0) = 1 - 1 = 0 \tag{76}$$

and

$$\frac{dG_c}{ds}(s=0) = -\frac{\sqrt{k^2 + 2\alpha y_f}}{ks + \alpha} exp(-st_f) + t_f \frac{s\sqrt{k^2 + 2\alpha y_f} + \alpha}{ks + \alpha} exp(-st_f) - k \frac{s\sqrt{k^2 + 2\alpha y_f} + \alpha}{(ks + \alpha)^2} exp(-st_f) = 0 \tag{77}$$

when t_f is given by Equation (61). It can again be noted that the transfer function given by Equation (75) cancels the poles and zeros of the second order system.

Figure 3 illustrates the variation of the maneuver time and the switch time as a function of varying maneuvers. The next section will generalize the design of pole-zero zero-pole canceling post-actuation controllers.

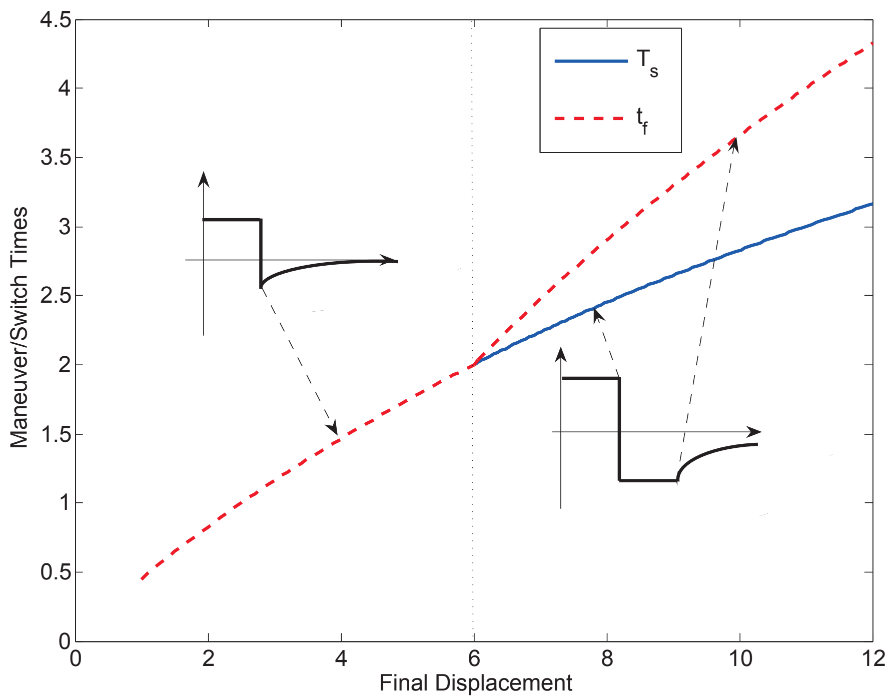

Fig. 3. Variation of Maneuver and Switch Time

3 Generalization

Consider a stable or marginally stable transfer function of the form

$$\frac{Y(s)}{U(s)} = G_p(s) = \frac{\sum_{i=0}^{m} a_i s^i}{s^n + \sum_{j=0}^{n-1} b_j s^j} \tag{78}$$

where $n \geq m$. All the zeros of the plant $G_p(s)$ are assumed to lie in the left-half of the complex plane. For systems which include non-minimum phase zeros, only the left-half plane zeros are considered in the design.

For the design of a post-actuation time-delay filter, consider the pre-filter parameterization:

$$\frac{U(s)}{R(s)} = G_c(s) = 1 + \sum_{k=1}^{L-1} (-1)^k 2\exp(-sT_k) + \exp(-sT_L) \frac{\sum_{r=0}^{m} c_r s^r}{\sum_{i=0}^{m} a_i s^i}. \tag{79}$$

The parameters of the time-delay filter, i.e., c_r, and T_k need to satisfy the constraints:

$$G_c(s = -p_j) = 0, \forall p_j = \text{roots}(s^n + \sum_{j=0}^{n-1} b_j s^j) \tag{80}$$

which guarantees cancelation of all the poles of the system with zeros of the time-delay filter. The poles include the rigid body poles, i.e., $s = 0, 0$. To ensure that the final values of the desired step input of magnitude y_f is achieved, we require:

$$y_f = \lim_{s \to 0} \frac{1}{s} s G_c G_p = \frac{a_0}{b_0} \left(\frac{c_0}{a_0} + 1 + \sum_{k=1}^{L-1} (-1)^k 2 \right). \tag{81}$$

If the limit is indeterminate, L'Hôpitals rule is used repeatedly until the limit can be determined. Finally, since the post-actuation control is required to satisfy the control constraints, constraints which ensure that the control bounds are satisfied after time T_L, which corresponds to the post-actuation phase of the control are added to the optimization problem. The post actuation part of the control is given by the equation:

$$U(s) = \frac{1}{s} \left(\pm 1 + \frac{\sum_{r=0}^{m} c_r s^r}{\sum_{i=0}^{m} a_i s^i} \right), \tag{82}$$

the inverse Laplace transform of which results in the solution $u(t)$. Since the zeros of the system which are the poles of the time-delay filter transfer function, are stable, one can ensure satisfaction of the control constraints by determining the time $t = T_{L+1}$ when:

$$\left. \frac{du}{dt} \right|_{T_{L+1}} = 0 \tag{83}$$

and requiring the control to be:

$$-1 \leq u(T_{L+1}) \leq 1. \tag{84}$$

An optimization problem can be posed to determine the parameters of the post-actuation time-delay filter. The statement of the problem is:

$$
\begin{aligned}
& \min \;\; J = T_L && \text{(85a)} \\
\text{subject to} \\
& 1+\sum_{k=1}^{L-1}(-1)^k 2\exp(-sT_k)+\exp(-sT_L)\frac{\sum_{r=0}^{m} c_r s^r}{\sum_{i=0}^{m} a_i s^i}\Bigg|_{s=-p_j} = 0 \forall p_j && \text{(85b)} \\
& \lim_{s\to 0}\frac{1}{s} sG_cG_p = \frac{a_0}{b_0}\left(\frac{c_0}{a_0}+1+\sum_{k=1}^{L-1}(-1)^k 2\right) = y_f && \text{(85c)} \\
& -1 \le u(T_{L+1}) = \mathcal{L}^{-1}\left(\frac{1}{s}\left(\pm 1+\frac{\sum_{r=0}^{m} c_r s^r}{\sum_{i=0}^{m} a_i s^i}\right)\right) \le 1 && \text{(85d)} \\
& \frac{du}{dt}\bigg|_{T_{L+1}} = 0 && \text{(85e)} \\
& T_{L+1} > T_L > T_{L-1} > \ldots > T_2 > T_1 > 0 && \text{(85f)}
\end{aligned}
$$

This is a nonlinear parameter optimization problem which can converge to multiple solutions. To ensure optimality, necessary conditions for optimality need to be derived.

In output transition problems, the goal is to force the output to the desired value and maintain it there for all future time. This can be accomplished using post-actuation control for system with minimum-phase zeros. The terminal output constraint is:

$$y(t_f) = Cx(t_f) = y_f. \tag{86}$$

To maintain the output at the desired value for all time greater than t_f, we require:

$$\frac{dy}{dt}\bigg|_{t_f} = C\left(Ax(t_f)+Bu(t_f)\right) = 0 \tag{87}$$

If $CB = 0$, then the next sequence of derivatives need to be tested until $u(t)$ explicitly shows up in the equation:

$$\frac{d^q y}{dt^q}\bigg|_{t_f} = CA^{q-1}\left(Ax(t_f)+Bu(t_f)\right) = 0, \tag{88}$$

which is called the q^{th} order state variable equality constraint. Here the q^{th} total time derivative is the control variable constraint which can be rewritten as inequality constraints using the limits of $u(t)$:

$$-CA^{q-1}\left(Ax(t_f)+B\right)) \le 0 \tag{89}$$

$$CA^{q-1}\left(Ax(t_f)-B\right)) \le 0. \tag{90}$$

The constrained optimal control problem can be represented as:

$$\min\ J = \int_0^{t_f} dt \tag{91a}$$
$$\text{subject to} \tag{91b}$$
$$\dot{x} = Ax + Bu \tag{91c}$$
$$x(0) = 0 \tag{91d}$$
$$Cx(t_f) = y_f \tag{91e}$$
$$CA^{q-1}x(t_f) = 0 \tag{91f}$$
$$-CA^{q-1}\left(Ax(t_f) + B\right)) \leq 0 \tag{91g}$$
$$CA^{q-1}\left(Ax(t_f) - B\right)) \leq 0 \tag{91h}$$
$$-1 \leq u(t) \leq 1 \tag{91i}$$

To determine the necessary conditions for optimality, the augmented cost function is:

$$\begin{aligned} J_a = {} & \nu_1 CA^{q-1}\left(Ax(t_f) - B\right)) + \nu_2 CA^{q-1}\left(-Ax(t_f) - B\right)) + \beta_0\left(Cx(t_f) - y_f\right) \\ & + \sum_{i=1}^{q-1} \beta_i CA^i x(t_f) + \int_0^{t_f}\left(1 + \lambda^T\left(Ax + Bu - \dot{x}\right)\right) dt \end{aligned} \tag{92}$$

where the Hamiltonian is defined as:

$$H = 1 + \lambda^T\left(Ax + Bu\right). \tag{93}$$

The necessary conditions for optimality are given by the equations:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = Ax + Bu \tag{94a}$$
$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -A^T\lambda \tag{94b}$$
$$u = -sign\left(B^T\lambda\right) \tag{94c}$$
$$x(0) = 0 \tag{94d}$$
$$\lambda(t_f) = (A^q)^T C^T \nu_1 - (A^q)^T C^T \nu_2 + C^T\beta_0 + \sum_{i=1}^{q-1}\beta_i\left(A^i\right)^T C^T \tag{94e}$$
$$\nu_1 \geq 0, \nu_2 \geq 0 \tag{94f}$$
$$y(t_f) = Cx(t_f) = y_f \tag{94g}$$
$$CA^{q-1}x(t_f) = 0 \tag{94h}$$
$$CA^q x(t_f) - CA^{q-1}B \leq 0 \tag{94i}$$
$$-CA^q x(t_f) - CA^{q-1}B \leq 0 \tag{94j}$$

The proposed time-optimal post-actuation controller design will be illustrated on the benchmark floating oscillator problem in the next section.

4 Benchmark Problem

For the two mass spring system shown in Figure 4, the transfer function relating the input to the displacement of the first mass is:

$$\frac{Y_1(s)}{U(s)} = \frac{s^2 + cs + 1}{s^2\left(s^2 + 2cs + 2\right)} \tag{95}$$

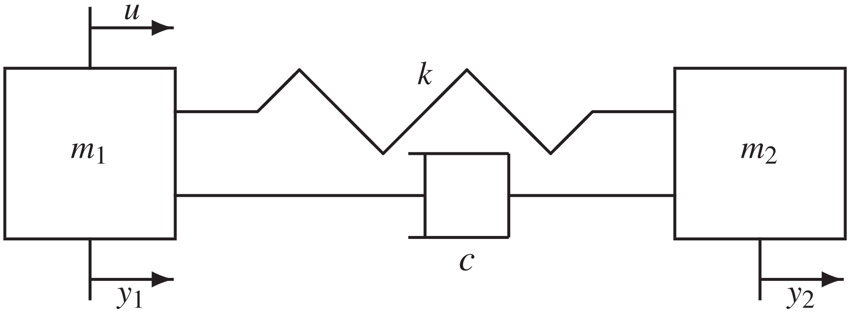

Fig. 4. Floating Oscillator

where the two masses are equal, the spring stiffness is unity and the damping constant is c. The unit step response of the system is:

$$y_1(t) = \frac{t^2}{4} + \frac{1}{4} - \frac{1}{4}e^{-ct}cos(\sqrt{2-c^2}t) - \frac{c}{4\sqrt{2-c^2}}e^{-ct}sin(\sqrt{2-c^2}t) \tag{96}$$

from which the higher derivatives can be calculated in closed form.

To determine the structure of the post actuation control, rewrite Equation (95) as a differential equation:

$$\ddddot{y}_1 + 2c\dddot{y}_1 + 2\ddot{y}_1 = \ddot{u} + c\dot{u} + u. \tag{97}$$

Laplace transform of Equation (97) leads to:

$$\begin{aligned}\left(s^4 + 2cs^3 + 2s^2\right)Y_1(s) - (s^3 y_1(0) + s^2\dot{y}_1(0) + s\ddot{y}_1(0) + \dddot{y}_1(0)) - 2c(s^2 y_1(0) + s\dot{y}_1(0) + \ddot{y}_1(0)) \\ -2(sy_1(0) + \dot{y}_1(0)) = \left(s^2 + cs + 1\right)U(s) - (su(0) + \dot{u}(0)) - cu(0).\end{aligned} \tag{98}$$

At the initiation of the post-actuation control, we have

$$y_1(0) = y_f, \dot{y}_1(0) = 0, \text{ or } Y_1(s) = \frac{y_f}{s} \tag{99}$$

which permits rewriting Equation (98) as

$$-(s\ddot{y}_1(0) + \dddot{y}_1(0)) - 2c\ddot{y}_1(0) = \left(s^2 + cs + 1\right)U(s) - (su(0) + \dot{u}(0)) - cu(0).$$

or

$$U(s) = \frac{(s+c)u(0) + \dot{u}(0) - (s+2c)\ddot{y}_1(0) - \dddot{y}_1(0)}{s^2 + cs + 1} = \frac{-as + b}{s^2 + cs + 1} \tag{100}$$

To arrive at a time-delay filter transfer function which generates the post-actuation control given by Equation (100), we have

$$\frac{1}{s}(-1 + G_{pa}) = \frac{(s+c)u(0) + \dot{u}(0) - (s+2c)\ddot{y}(0) - \dddot{y}(0)}{s^2 + cs + 1} \tag{101}$$

assuming the control magnitude is -1 prior to the initiation of the post-actuation control. Solving Equation (101) for G_{pa}, we have

$$G_{pa} = \frac{s(s+c)u(0) + s\dot{u}(0) - s(s+2c)\ddot{y}_1(0) - s\dddot{y}_1(0)}{s^2 + cs + 1} + 1 \tag{102}$$

which can be simplified to:

$$G_{pa} = \frac{s^2\overbrace{(u(0) - \ddot{y}_1(0) + 1)}^{P} + s\overbrace{(cu(0) + \dot{u}(0) - 2c\ddot{y}_1(0) - \dddot{y}_1(0))}^{Q} + 1}{s^2 + cs + 1} \tag{103}$$

Assuming a three switch time-delay filter corresponds to the time-optimal structure for the output-to-output transition control profile, the time-delay filter's transfer function can be parameterized as:

$$G_c = \left(1 - 2e^{-sT_1} + 2e^{-sT_2} - 2e^{-sT_3} + \frac{Ps^2 + Qs + 1}{s^2 + cs + 1} e^{-sT_4}\right) \tag{104}$$

The post actuation control is given by the step response of system with the transfer function given by Equation (103) which can be represented as:

$$u(t) = 1 + 2\,\frac{\left(b + \frac{ac}{2}\right) \mathrm{e}^{-\frac{c}{2}t} \sin\left(1/2\sqrt{-c^2 + 4}t\right)}{\sqrt{-c^2 + 4}} - a\mathrm{e}^{-\frac{c}{2}t} \cos\left(1/2\sqrt{-c^2 + 4}t\right) \tag{105}$$

where

$$a = Pc - Q, \text{ and } b = c - P. \tag{106}$$

The derivative of the post actuation control

$$\begin{aligned}\dot{u}(t) = &-\frac{\left(b + \frac{ac}{2}\right) c\mathrm{e}^{-1/2tc} \sin\left(1/2\sqrt{-c^2+4}t\right)}{\sqrt{-c^2+4}} + \left(b + \frac{ac}{2}\right) \mathrm{e}^{-1/2tc} \cos\left(1/2\sqrt{-c^2 + 4}t\right) + \\ &\frac{ac}{2}\mathrm{e}^{-1/2tc} \cos\left(1/2\sqrt{-c^2 + 4}t\right) + 1/2a\mathrm{e}^{-1/2tc} \sin\left(1/2\sqrt{-c^2 + 4}t\right) \sqrt{-c^2 + 4}\end{aligned} \tag{107}$$

can be used to determine when $u(t)$ reaches it minimum and maximum. We can now constrain the limiting values of $u(t)$ to be equal to ± 1 to satisfy the bounds on the control. For the post actuation problem, the additional constraints which need to be imposed to ensure that the post-actuation control does not violate the control constraints requires solving the equations:

$$\dot{u}(T_5 - T_4) = 0 \tag{108}$$

$$u(T_5 - T_4) = -1 \tag{109}$$

since $u(T_5 - T_4) = 0$ corresponds to the minimum of $u(t)$ which when included as part of the time-delay filter response given by Equation (104) corresponds to $u(T_5) = -1$.

4.1 Necessary Conditions for Optimality

The state space model of the benchmark problem with the output being the displacement of the mass acted on by the input is:

$$\underbrace{\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{B} u \tag{110}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}}_{x} \tag{111}$$

The costate equation is:

$$\underbrace{\begin{Bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \\ \dot{\lambda}_4 \end{Bmatrix}}_{\dot{\Lambda}} = -\underbrace{\begin{bmatrix} 0 & 0 & -k & k \\ 0 & 0 & k & -k \\ 1 & 0 & -c & c \\ 0 & 1 & c & -c \end{bmatrix}}_{A}\underbrace{\begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{Bmatrix}}_{\Lambda} \tag{112}$$

Since the Hamiltonian is:

$$H = 1 + \Lambda^T (Ax + Bu), \tag{113}$$

the time-optimal control is given by the equation:

$$u = -sign\left(\Lambda^T B\right). \tag{114}$$

The terminal constraints are:

$$y(t_f) = y_f = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = x_1(t_f) \tag{115}$$

$$\dot{y}(t_f) = 0 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = x_3(t_f) \tag{116}$$

$$\ddot{y}(t_f) = 0 = \begin{bmatrix} -k & k & -c & c \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + u(t_f) \tag{117}$$

The constraint given by Equation (117) can be rewritten as inequality constraints:

$$\begin{bmatrix} -k & k & -c & c \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} - 1 \leq 0 \tag{118}$$

$$-\begin{bmatrix} -k & k & -c & c \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} - 1 \leq 0 \tag{119}$$

The augmented cost function includes the following terms:

$$\begin{aligned} &\nu_1 \left(\begin{bmatrix} -k & k & -c & c \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} - 1 \right) + \nu_2 \left(-\begin{bmatrix} -k & k & -c & c \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} - 1 \right) \\ &+\beta_1 (x_1(tf) - y_f) + \beta_2 x_3(tf) \end{aligned} \tag{120}$$

which leads to the terminal costate constraint:

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{Bmatrix}(t_f) = \begin{bmatrix} -k \\ k \\ -c \\ c \end{bmatrix} \nu_1 - \begin{bmatrix} -k \\ k \\ -c \\ c \end{bmatrix} \nu_2 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \beta_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \beta_2 \tag{121}$$

When the inequality constraints Equation (118)-(119) are inactive, ν_1 and ν_2 are both 0. The terminal value of the control can then be determined from Equation (117) resulting in:

$$u(t_f) = -\begin{bmatrix} -k & k & -c & c \end{bmatrix} \begin{Bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \\ x_4(t_f) \end{Bmatrix}. \tag{122}$$

Since the Hamiltonian is 0 at the terminal time, we have:

$$\begin{Bmatrix} \lambda_1(t_f) & \lambda_2(t_f) & \lambda_3(t_f) & \lambda_4(t_f) \end{Bmatrix} \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix} \begin{Bmatrix} x_1(t_f) \\ x_2(t_f) \\ x_3(t_f) \\ x_4(t_f) \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t_f) \right) = -1. \tag{123}$$

Equations (110),(112), (115), (116), (118), (119) and (121) form the necessary conditions for optimality.

4.2 Large Maneuver

Parameterize the time-optimal post-actuation control as a three switch bang-bang control followed by a decaying sinusoidal post-actuation control as shown in Figure (5). Note that since the zeros of the system which need to be cancelled are a pair of complex left-half plane zeros, the poles of the time-delay filter which cancel them will result in a damped harmonic post-actuation control profile.

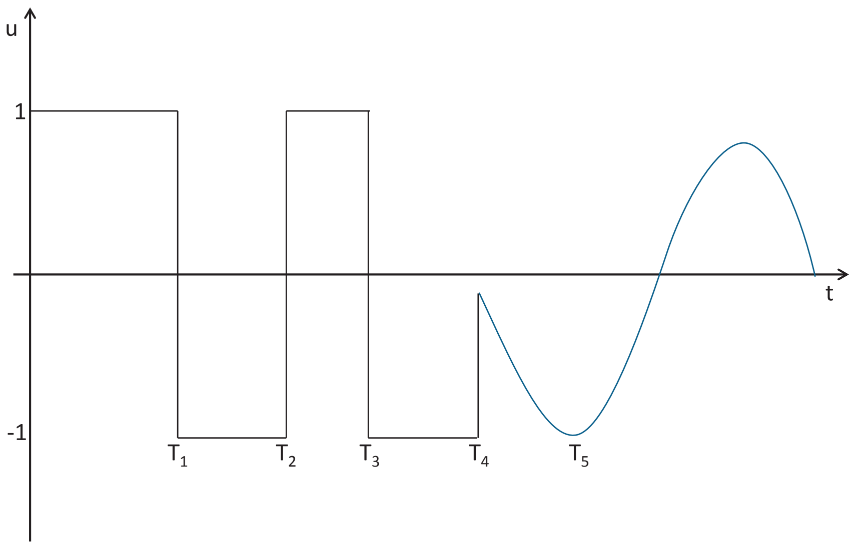

Fig. 5. Three-Switch Post-Actuation Control Profile

The time-delay filter which generates the control profile illustrated in Figure (5) is:

$$G_c = \left(1 - 2e^{-sT_1} + 2e^{-sT_2} - 2e^{-sT_3} + \frac{Ps^2 + Qs + 1}{s^2 + cs + 1} e^{-sT_4} \right). \tag{124}$$

The zeros of the transfer function $G_c(s)$ should cancel the poles of the system located at:

$$s = 0, 0, -c \pm i\sqrt{2-c^2}. \tag{125}$$

The parameter optimization problem can be posed as:

$$\begin{aligned}
&\min \;\; J = T_4 && \text{(126a)}\\
\text{subject to} & && \text{(126b)}\\
&\frac{dG_c}{ds}(s=0) = 2T_1 - 2T_2 + 2T_3 - T_4 + Q - c = 0 && \text{(126c)}\\
&1 + \sum_{k=1}^{3}(-1)^k 2e^{-cT_k} cos\sqrt{2-c^2}T_k + (c^2 - Qc - 1 + 2P)e^{-cT_4} cos\sqrt{2-c^2}T_4 &&\\
&+\sqrt{2-c^2}(c-Q)e^{-cT_4} sin\sqrt{2-c^2}T_4 = 0 && \text{(126d)}\\
&\sum_{k=1}^{3}(-1)^k 2e^{-cT_k} sin\sqrt{2-c^2}T_k - (c^2 - Qc - 1 + 2P)e^{-cT_4} sin\sqrt{2-c^2}T_4 &&\\
&+\sqrt{2-c^2}(c-Q)e^{-cT_4} cos\sqrt{2-c^2}T_4 = 0 && \text{(126e)}\\
&(P - QT_4 + \frac{T_4^2}{2} - 1 + c(T_1 - T_2 + T_3)) + \sum_{k=1}^{3}(-1)^k T_k^2 = 2y_f && \text{(126f)}\\
&\dot{u}(T_5 - T_4) = 0 && \text{(126g)}\\
&u(T_5 - T_4) = -1 && \text{(126h)}\\
&0 < T_1 < T_2 < T_3 < T_4 < T_5 && \text{(126i)}
\end{aligned}$$

where $u(t)$ and $\dot{u}(t)$ are given by Equations (105) and (107).

For an output displacement of $y_f = 3$, Figure 6 illustrates the post-actuation time-optimal control, the switching curve and the corresponding evolution of the displacement of the first mass of the benchmark floating oscillator. It can be noted that the post-actuation control profile is a damped harmonic which reaches the minimum of -1 at 6.2 seconds.

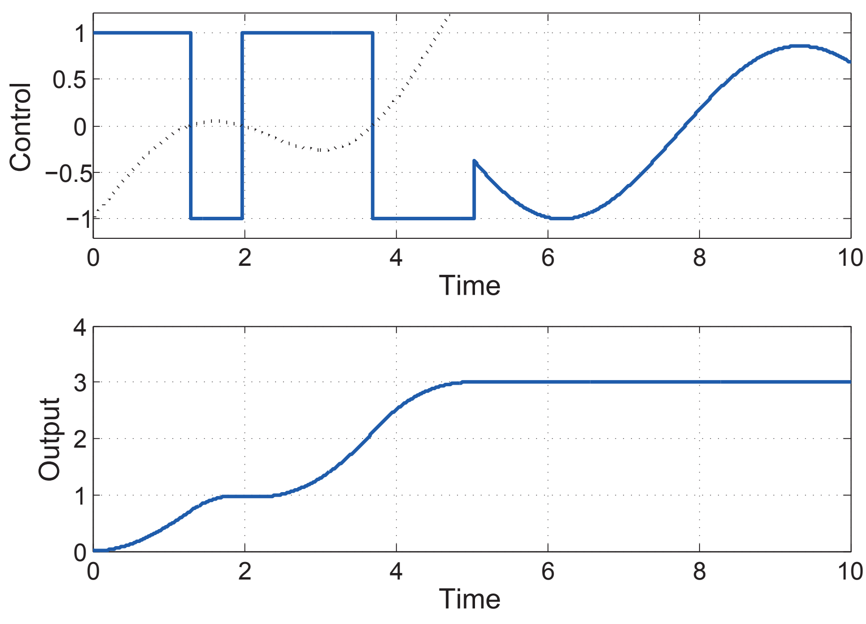

Fig. 6. Large Maneuver Three-Switch Post-Actuation

Figure 7 illustrates the variation of the switch times and the maneuver time as a function of the final displacement y_f. It can be seen that the first two switches shown by the dash and dotted line collapse in the vicinity of $y_f = 1.29$. The parameterization presented in this section corresponds to all maneuvers greater than $y_f = 1.29$. For maneuvers smaller than $y_f = 1.29$, a new parameterization of the control profile is necessary which is presented in the next section.

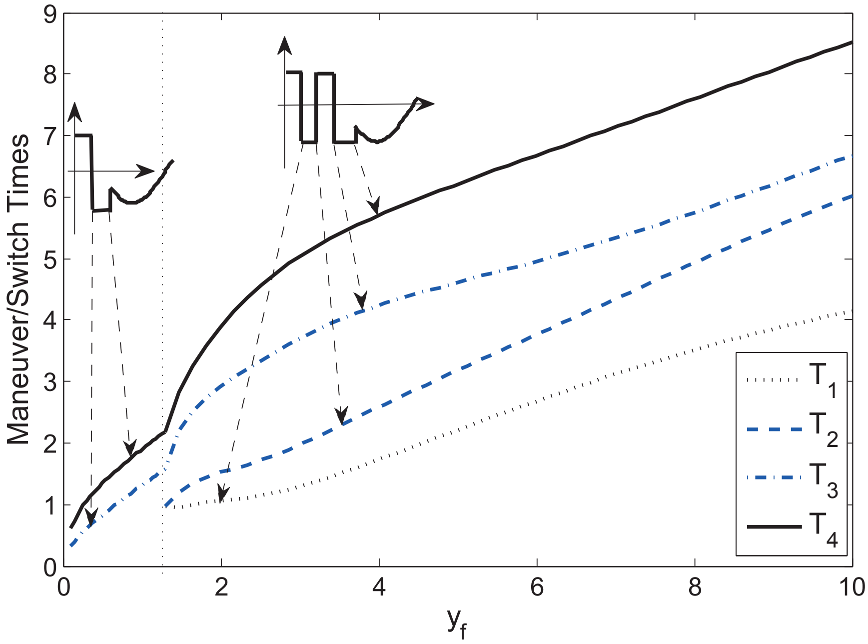

Fig. 7. Switch and Maneuver Time

4.3 Small Maneuver

Parameterize the time-optimal post-actuation control as a single switch bang-bang control followed by a decaying sinusoidal post-actuation control as shown in Figure 8.

The time-delay filter which generates the control profile illustrated in Figure 8 is:

$$G_c = \left(1 - 2e^{-sT_1} + \frac{Ps^2 + Qs + 1}{s^2 + cs + 1} e^{-sT_2}\right). \tag{127}$$

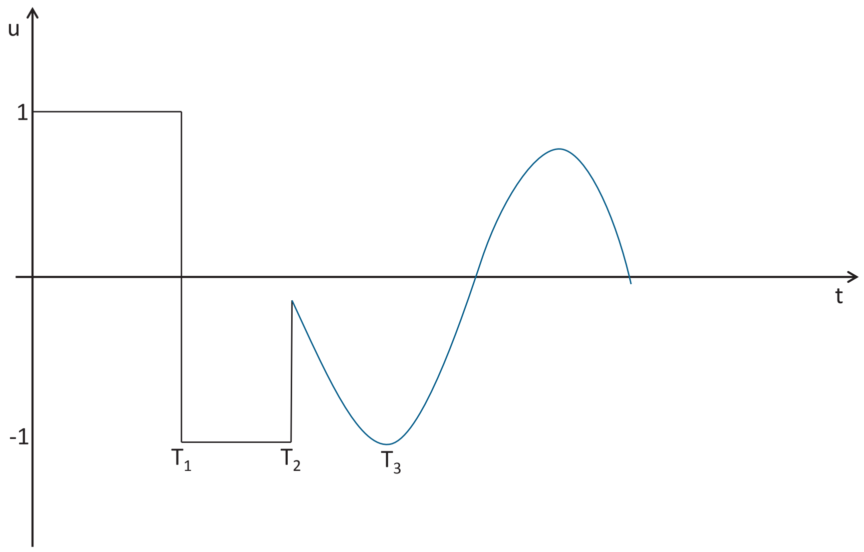

Fig. 8. Single-Switch Post-Actuation Control Profile

The parameter optimization problem can be posed as:

$$\min \ J = T_2 \tag{128a}$$

subject to (128b)

$$\frac{dG_c}{ds}(s=0) = 2T_1 - T_2 + Q - c = 0 \tag{128c}$$

$$1 - 2e^{-cT_1}cos\sqrt{2-c^2}T_1 + (c^2 - Qc - 1 + 2P)e^{-cT_2}cos\sqrt{2-c^2}T_2 + \sqrt{2-c^2}(c-Q)e^{-cT_2}sin\sqrt{2-c^2}T_2 = 0 \tag{128d}$$

$$-2e^{-cT_1}sin\sqrt{2-c^2}T_1 - (c^2 - Qc - 1 + 2P)e^{-cT_2}sin\sqrt{2-c^2}T_2 + \sqrt{2-c^2}(c-Q)e^{-cT_2}cos\sqrt{2-c^2}T_2 = 0 \tag{128e}$$

$$(P - QT_2 + \frac{T_2^2}{2} - 1 + cT_1) - 1T_1^2 = 2y_f \tag{128f}$$

$$\dot{u}(T_3 - T_2) = 0 \tag{128g}$$

$$u(T_3 - T_2) < -1 \tag{128h}$$

$$0 < T_1 < T_2 < T_3 \tag{128i}$$

Figure 7 illustrates the variation of the switch times and the maneuver time as a function of the final displacement y_f for maneuvers smaller than $y_f = 1.29$.

Figure 9 illustrates a typical post-actuation time-optimal control profile. For a maneuver of 1 unit, it can be seen that a single switch bang-bang control transitions the output from an initial state of rest to a terminal state of rest. The post-actuation control as opposed to the large maneuver case does not reach the control limits in the post-actuation phase of the control.

Figure 10 illustrates the variation in the maneuver time of the state-to-state transition time optimal control (solid line) and the time optimal output transition control (dashed line), as a function of the maneuver y_f. It can be seen that the post-actuation time-optimal control consistently requires significantly smaller time to complete the maneuver.

It should be noted that the number of switches for the time optimal control cannot, in most application be prescribed at the outset. The number of switches as shown in Figures 3 and 7 changes as a function of the maneuver. They also change for the same maneuver, as a function of varying damping coefficient as shown by Singh [23] for minimum time control. Consequently, the designer must select a certain number of switches and solve the parameter optimization problem and then check if all the necessary conditions of optimality are satisfied. If they are not, a different parametrization is selected and the process is repeated till the right number of switches which satisfy all the necessary conditions of optimality is identified.

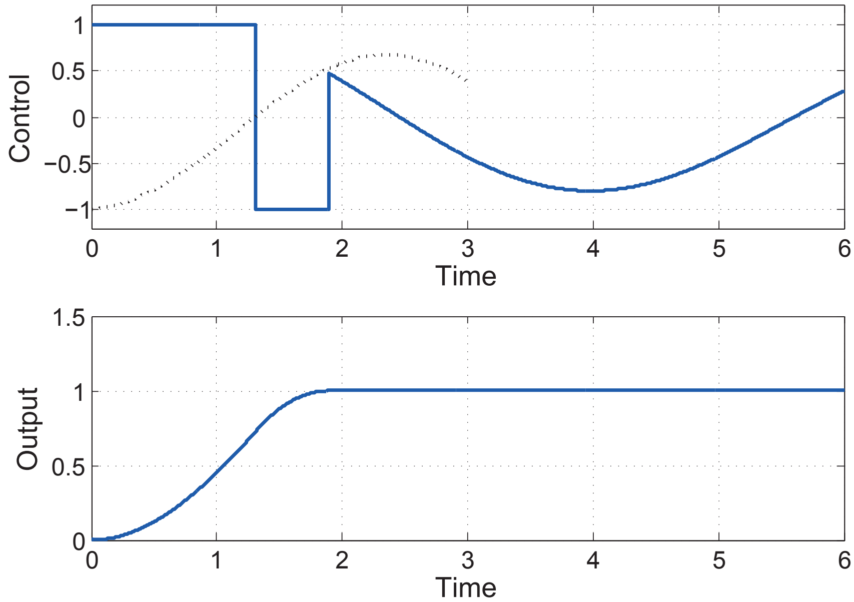

Fig. 9. Small Maneuver Single-Switch Post-Actuation

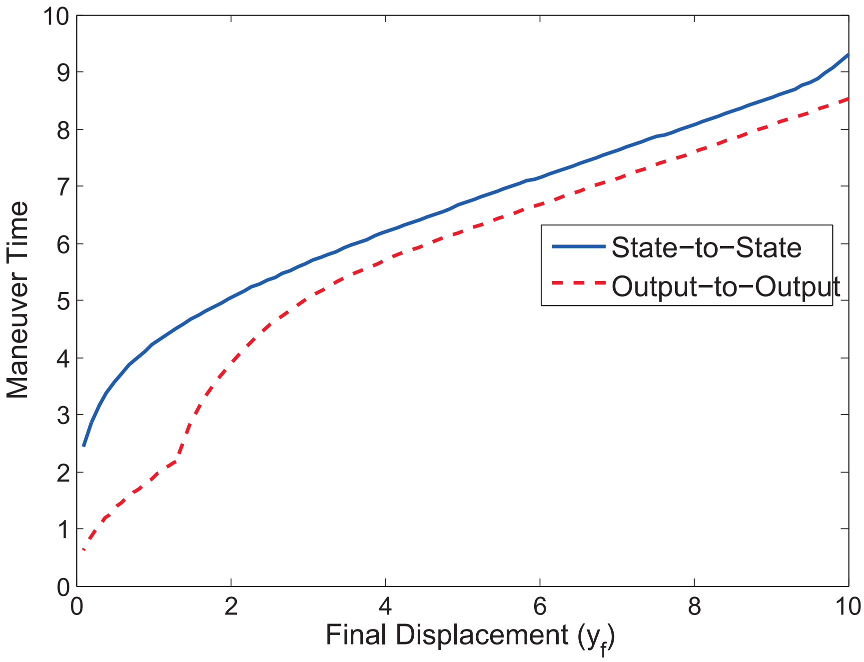

Fig. 10. Comparison of Maneuver Times

5 Conclusions

An analytical solution to the post-actuation time-optimal controller for a double integrator with a left-half plane zero is presented to motivate the structure of the post-actuation control profile. A corresponding frequency domain design approach is shown to equate to cancelling the poles and zeros of the system transfer function with the zeros and poles of a transfer function of a time-delay filter. The transition of the structure of the time-optimal control profile as a function of maneuver distance is illustrated for the double integrator problem. The post-actuation time-optimal control profile is generalized and illustrated on the benchmark floating oscillator problem where it is shown that the post-actuation control profile is a damped harmonic. The transition phase of the control is shown to be bang-bang with a three switch structure for large maneuvers and a single switch structure for smaller maneuvers. Comparison with the time-optimal state-to-state transition for the benchmark problem helps illustrate the reduction in maneuver time of the output transition time-optimal control. For systems with non-minimum phase zeros, a pre-actuation phase in conjunction with a bang-bang transition phase is the solution to the time-optimal output transition problem. The proposed post-actuation design approach can be used to solve the pre-actuation design by reversing time and solving for the transformed post-actuation time-optimal controller. To desensitize the performance of the post-actuation control to uncertainties in the location of the system poles, multiples zeros of the time-delay filter can be located at the nominal location of the uncertain poles. When knowledge of the support of the uncertain poles are known, a minimax optimization problem can be formulated to minimize the worst performance of the controller over the domain of uncertainty.

Acknowledgements

The authors gratefully acknowledge helpful discussions with Prof. S. R. Vadali of Aerospace Engineering, Texas A & M University.

References

[1] Devasia, S., 2012. "Time-optimal control with pre/post actuation for dual-stage systems". *Control Systems Technology, IEEE Transactions on,* **20**(2), march, pp. 323 –334.

[2] Stearns, H., Yu, S., Fine, B., Mishra, S., and Tomizuka, M., 2008. "A comparative study of feedforward tuning methods for wafer scanning systems". *ASME Conference Proceedings,* **2008**(43352), pp. 669–676.

[3] Singhose, W., and Vaughan, J., 2011. "Reducing vibration by digital filtering and input shaping". *Control Systems Technology, IEEE Transactions on,* **19**(6), nov., pp. 1410 –1420.

[4] Singh, T., and Vadali, S. R., 1993. "Input-shaped control of three-dimensional maneuvers of flexible spacecraft". *Journal of Guidance, Control and Dynamics,* **16**(6), Nov., pp. 1061–1068.

[5] Singer, N., and Seering, W. P., 1990. "Pre-shaping command inputs to reduce system vibration". *Journal of Dynamic Systems, Measurement and Control,* **112**(1), pp. 76–82.

[6] Singh, T., and Vadali, S. R., 1993. "Robust time delay control". *Journal of Dynamic Systems, Measurement and Control,* **115**(2), pp. 303–306.

[7] Singhose, W., Derezinski, S., and Singer, N., 1996. "Extra-insensitive input shapers for controlling flexible spacecraft". *Journal of Guidance, Control and Dynamics,* **19**(2), pp. 385–391.

[8] Singh, T., 2002. "Minimax design of robust controllers for flexible systems". *Journal of Guidance, Control and Dynamics,* **25**(5), pp. 868–875.

[9] Singh, T., 2004. "Jerk limited input shapers". *ASME Journal of Dynamic Systems, Measurement and Control,* **126**(1), Mar., pp. 215–219.

[10] Junkins, J. L., Rahman, Z. H., and Bang, H., 1991. "Near-minimum-time maneuvers of flexible structures by parameter optimization". *Journal of Guidance, Control and Dynamics,* **14**(2), pp. 406–415.

[11] Dijkstra, B., and Bosgra, O., 2003. "Exploiting iterative learning control for input shaping, with application to a wafer stage". In American Control Conference, 2003. Proceedings of the 2003, Vol. 6, pp. 4811 – 4815 vol.6.

[12] Bodson, M., 1997. "An adaptive algorithm for the tuning of two input shaping methods". In American Control Conference, 1997. Proceedings of the 1997, Vol. 3, pp. 1340 –1344 vol.3.

[13] Singh, G., Kabamba, P. T., and McClamroch, N. H., 1989. "Planar time-optimal control, rest-to-rest slewing of flexible spacecraft". *Journal of Guidance, Control and Dynamics,* **12**(1), pp. 71–81.

[14] Ben-Asher, J., Burns, J. A., and Cliff, E. M., 1992. "Time-optimal slewing of flexible spacecraft". *Journal of Guidance, Control and Dynamics,* **15**(2), pp. 360–367.

[15] Singh, T., and Vadali, S. R., 1994. "Robust time-optimal control: Frequency domain approach". *Journal of Guidance, Control and Dynamics,* **17**(2), pp. 346–353.

[16] Wie, B., Sinha, R., Sunkel, J., and Cox, K., 1993. "Robust fuel- and time-optimal control of uncertain flexible space strcutures". In Guidance, Dynamics and Control Conference.

[17] Singh, T., 1995. "Fuel/time optimal control of the benchmark problem". *Journal of Guidance, Control and Dynamics,* **18**(6), pp. 1225–1231.

[18] Muenchhof, M., and Singh, T., 2003. "Jerk limited time optimal control of flexible structures". *ASME Journal of Dynamic Systems, Measurement and Control,* **125**(1), pp. 139–142.

[19] Ben-Itzak, S., and Karniel, A., 2008. "Minimum acceleration criterion with constraints implies bang-bang control as an underlying principle for optimal trajectories of arm reaching movements". *Neural Computation,* **20**, pp. 779–812.

[20] Iamratanakul, D., Jordan, B., Leang, K., and Devasia, S., 2008. "Optimal output transitions for dual-stage systems". *Control Systems Technology, IEEE Transactions on,* **16**(5), sept., pp. 869 –881.

[21] Iamratanakul, D., and Devasia, S., 2008. "Feedforward input design for minimum-time/energy, output transitions for dual-stage systems". In American Control Conference, 2008, pp. 3263 –3268.

[22] Devasia, S., 2012. "Time-optimal control with pre/post actuation for dual-stage systems". *Control Systems Technology, IEEE Transactions on,* **20**(2), march, pp. 323 –334.

[23] Singh, T., 1996. "Effect of damping on the structure of time-optimal controllers". *Journal of Guidance, Control and Dynamics,* **19**(5), Sept., pp. 1182–1184.