International Journal of Systems Science
Vol. 36, No. 10, 15 August 2005, 651–662

Taylor & Francis
Taylor & Francis Group

Equilibrium stability in decentralized design systems

VINCENT CHANRON, TARUNRAJ SINGH and KEMPER LEWIS*

Department of Mechanical and Aerospace Engineering,
University at Buffalo – State University of New York, Buffalo, NY 14260, USA

(*Received 22 April 2004; in final form 9 June 2005*)

The focus of this paper is on complex systems, and it presents a theoretical study of the design of complex engineering systems. More particularly, this paper studies the stability of equilibria in decentralized design environments. Indeed, the decentralization of decisions is often recommended in the design of complex systems, and the decomposition and coordination of decisions are a great challenge. The mechanisms behind this network of decentralized design decisions create difficult management and coordination issues. However, developing efficient design processes is paramount, especially with market pressures and customer expectations. Standard techniques to modelling and solving decentralized design problems typically fail to understand the underlying dynamics of the decentralized processes and therefore result in sub-optimal solutions. This paper aims to model and understand the mechanisms and dynamics behind a decentralized set of decisions within a complex design process. Complex systems that are multidisciplinary and highly nonlinear in nature are the primary focus of this paper. Therefore, techniques such as response surface approximations and Game Theory are used to discuss and solve the issues related to multidisciplinary optimization. Nonlinear control theory is used in this paper as a new approach to study the stability of equilibrium points of the design space. Illustrations of the results are provided in the form of the study of the decentralized design of a pressure vessel.

Keywords: Decentralized design; Decomposition; Game Theory; Nash equilibrium; Response surface; Nonlinear control; Lyapunov theory

1. Introduction

The focus of this paper is the design of complex engineering systems, or those systems that necessitate the decomposition into smaller subsystems in order to reduce the complexity of the design problems. Most of these systems are very large and multidisciplinary in nature, and therefore have a great number of subsystems and components. This section presents a literature review of the area of engineering design, and more particularly of distributed design.

Over the past years, many different techniques, algorithms, software, and methods have been proposed to solve these design problems. They mainly concentrate on finding a solution which is optimal in some sense, while keeping the computational cost as low as possible (Guarino Lo Bianco and Piazzi 2001). One of these methods, known as *decomposition*, is now seen as a necessary step in design. Indeed, these complex engineering systems are multidisciplinary in nature, and it is therefore impossible for one designer, or even a single design team, to consider the entire system as a single design problem. Typically, in complex systems, breaking it up into smaller units or *subsystems* will make the system more manageable (Kusiak and Wang 1993, Krishnamachari and Papalambros 1997).

The decentralization of decisions is unavoidable in a large organization where having only one centralized decision-maker is usually not applicable (Lee and Whang 1999). A more effective way is to delegate

*Corresponding author. Email: kelewis@eng.buffalo.edu

International Journal of Systems Science
ISSN 0020–7721 print/ISSN 1464–5319 online © 2005 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00207720500219963

decision responsibilities to the appropriate person, team, or supplier. In fact, decentralization is recommended as a way to speed up product-development processes and decrease the computational time and the complexity of the problem (Prewitt 1998).

While the decomposition of complex problems certainly creates a series of smaller, less complex problems, it also creates several challenging issues associated with the coordination of these less complex problems. The origin of these problems is the fact that the less complex subproblems are usually coupled. Systems are said to be coupled if their solution is dependent upon information from other subproblems. The ideal case would be a system that could be broken up into subsystems without interdependence. Unfortunately, design variables and parameters usually have an influence on several subproblems. Design variables and parameters that are controlled within a subsystem are called *local*, while *non-local* information is controlled by another subsystem. A formal definition and discussion of coupled subsystems can be found in Balling and Sobiezczanski-Sobieski (1994).

Previous work has been done on the decomposition of the system into smaller ones using Design Structure Matrices (McCulley and Bloebaum 1996), or a hierarchical approach (Sobieszczanski-Sobieski *et al.* 1985), or by effectively propagating the desirable top-level design specifications to appropriate subsystems (Michelena *et al.* 2002). The efficiency of these decomposition schemes has also been compared (Braun *et al.* 1996).

Also, previous work has concentrated on solving coupled design problems with interacting subsystems using Game Theory. The main goal is to try to improve the quality of the final solution in a multiobjective, distributed design optimization problem (Vincent 1983). Previous work in Game Theory includes work to model the interactions between the designers if several design variables are shared among designers (Lewis and Mistree 1997). In Marston and Mistree (2000), Game Theory is formally presented as a method to help designers make strategic decisions in a scientific way. In Hernandez *et al.* (2002), distributed collaborative design is viewed as a non-cooperative game, and maintenance considerations are introduced into a design problem using concepts from Game Theory. In Allen (2001), the manufacturability of multi-agent process planning systems is studied using Game Theory concepts. In Rao *et al.* (1997), non-cooperative protocols are studied, and the application of Stackelberg leader/follower solutions is shown. The distinction between cooperative and non-cooperative scenarios is made using the notion of cooperation degree in order to characterize competitive situations quantitatively (Xu 1999). Also, in Chen and Li (2001), a Game Theory approach is used to address and describe a multifunctional team approach for concurrent parametric design. This set of previous work has established a solid foundation for the application of game theory in design, but has not directly studied the mechanisms of convergence in a generic decentralized design problem.

Studying the convergence of such processes is paramount, and this is done by formally describing the dynamics and interactions involved in such design scenarios. The dynamics involved in distributed design have been studied using a mathematical model (Loch *et al.* 2003) or a computer-based system (Whitfield *et al.* 2002). The concept of stability has been introduced in the computer-based management of distributed system (Lee and Ghosh 2000), and the first rigorous results for convergence in distributed design have been presented in Chanron and Lewis (2003), where convergence criteria for simple quadratic decentralized problems involving two designers were developed. This paper builds on these results to extend convergence criteria to more complex problems. The convergence of these types of complex problems has not been studied, and to date, no closed-form solution has been found, leaving room for fundamental work. The main contribution of this paper is to propose a mathematically rigorous approach to address these problems.

Lyapunov theory has been extensively used to study the convergence and stability of control systems (Fink and Singh 1998, Xu 2002), which are similar in nature to distributed design problems. The novelty of the work in this paper is in exploiting concepts from nonlinear control theory to decentralized design scenarios in order to study the stability of some highly nonlinear equilibriums. The next section describes how we propose to investigate these scenarios using these new techniques.

2. Design scenarios

In this section, the main game theory scenarios used to solve large multiobjective design problems are reviewed and discussed. We assume that the design problem has already been subdivided into smaller subsystems, either naturally because several different companies interact on the design of the same product, or because the system has been subdivided into smaller subsystems using one of the techniques described in the previous section. A good description of the different scenarios in design can be found in Rao *et al.* (1997) and Lewis and Mistree (1998).

As mentioned in the previous section, Game Theory is usually used as a way to study these design scenarios. Table 1 presents the Game-Theoretic formulation for an optimization design problem with two designers

Table 1. Multi-player optimization problem formulation.

Player 1's model	Player 2's model
Minimize	Minimize
$\mathbf{F}_1(\mathbf{x}_1, \mathbf{x}_{2c}) = \{F_1^1, F_1^2, \ldots, F_1^p\}$	$\mathbf{F}_2(\mathbf{x}_2, \mathbf{x}_{1c}) = \{F_2^1, F_2^2, \ldots, F_2^q\}$
subject to	subject to
$g_j^1(\mathbf{x}_1, \mathbf{x}_{2c}) \le 0 \quad j = 1, \ldots, m_1$	$g_j^2(\mathbf{x}_2, \mathbf{x}_{1c}) \le 0 \quad j = 1, \ldots, m_2$
$h_k^1(\mathbf{x}_1, \mathbf{x}_{2c}) = 0 \quad k = 1, \ldots, l_1$	$h_k^2(\mathbf{x}_2, \mathbf{x}_{1c}) = 0 \quad k = 1, \ldots, l_2$
$\mathbf{x}_{1L} \le \mathbf{x}_1 \le \mathbf{x}_{1U}$	$\mathbf{x}_{2L} \le \mathbf{x}_2 \le \mathbf{x}_{2U}$

(also called *players*). In this table, $\mathbf{x}_1$ represents the vector of design variables controlled by designer 1, while designer 2 controls design variable vector $\mathbf{x}_2$. We denote $\mathbf{x}_{1c}$ and $\mathbf{x}_{2c}$ as the *non-local* design variables, or variables that appear in a model but are controlled by the other player.

A complete description of all the protocols can be found in Lewis and Mistree (1998), but we present here only the three main types.

2.1. *Cooperative protocol*

In this protocol, both players have knowledge of the other player's information, and they work together to find a Pareto solution. A pair $(\mathbf{x}_{1P}, \mathbf{x}_{2P})$ is Pareto optimal (Pareto 1906) if no other pair $(\mathbf{x}_1, \mathbf{x}_2)$ exists, such that

$$F_i(\mathbf{x}_1, \mathbf{x}_2) \le F_i(\mathbf{x}_{1p}, \mathbf{x}_{2p}) \quad i = 1, 2$$

$$\text{and} \quad F_j(\mathbf{x}_1, \mathbf{x}_2) < F_j(\mathbf{x}_{1p}, \mathbf{x}_{2p}) \quad \text{for at least one } j = 1, 2. \tag{1}$$

2.2. *Non-cooperative protocol*

This protocol occurs when full coalition among players is not possible due to organizational, information, or process barriers. Players must make decisions by assuming the choices of the other decision-makers. In an iterative approach, the final solution would be a Nash equilibrium. A strategy pair ($\mathbf{x}_{1N}$, $\mathbf{x}_{2N}$) is a Nash solution if

$$\begin{aligned} F_1(x_{1N}, x_{2N}) &= \min_{x_1} F_1(x_1, x_{2N}) \\ \text{and} \quad F_2(x_{1N}, x_{2N}) &= \min_{x_2} F_2(x_{1N}, x_2). \end{aligned} \tag{2}$$

This solution has the property of being individually stable but is not necessarily collectively stable. The Nash equilibrium also has the property of being the fixed point of two subsets of the feasible space:

$$(x_{1N}, x_{2N}) \in X_{1N}(x_{2N}) \times X_{2N}(x_{1N}) \tag{3}$$

where

$$\begin{cases} X_{1N}(x_2) = \{x_{1N} \mid F_1(x_{1N}, x_2) = \min_{x_1} F_1(x_1, x_2)\} \\ X_{2N}(x_1) = \{x_{2N} \mid F_2(x_1, x_{2N}) = \min_{x_2} F_2(x_1, x_2)\}. \end{cases}$$

$X_{1N}(x_2)$ and $X_{2N}(x_1)$ are called the *Rational Reaction Sets* of the two players. The Rational Reaction Set (RRS) of a player is a function that embodies their reactions to decisions made by other players.

2.3. *Leader/Follower protocol*

When one player dominates another, they have a leader/follower relationship. This is a common occurrence in a design process when one discipline dominates the design (when one discipline plays a large role), or in a design process that involves a sequential execution of interrelated disciplinary processes. Player 1 is said to be the leader if they declare their strategy first, by assuming that Player 2 behaves rationally. Thus, the model of Player 1 as a leader is as follows:

$$\begin{aligned} \text{Minimize} \quad & F_1(x_1, x_2) \\ \text{subject to} \quad & x_2 \in X_{2N}(x_1), \end{aligned} \tag{4}$$

where $X_{2N}(x_1)$ is the RRS of player 2.

The focus of this paper is design in decentralized environments. In that case, even within the same corporation, perfect information and cooperation are difficult to achieve due to several factors, including the complexity of the design, geographic separation or information privacy, thus leading to limited cooperation. Therefore, we focus on non-cooperative relationships between designers. In other words, we focus on decentralized design scenarios where full and efficient exchange of all information among subsystems is not possible.

In these environments, the goal is to determine whether the Nash equilibrium is collectively stable or not. This issue has never been investigated, although its relevance is noted in several papers (Vincent 1983, Marston and Mistree 2000). This issue of an unstable equilibrium is challenging. Indeed, in the case of instability, designers will never agree on a final design, since one of the designers will always be able to change the value of their design variables and improve their objective function. In this case, the process by which the two designers might go about choosing the final design is then difficult to predict, but in the absence of any additional information or intervention by a third party, choosing the final design will be problematic. The first steps towards the study of stability have been laid out for simple design problems with quadratic objective functions (Chanron and Lewis 2003). The focus of this paper is to study those properties for more complicated and non-linear problems. The next section presents the steps to follow for such a study.

3. Equilibriums of the design space

In this section, we describe how to obtain the equilibrium points of a specified design space, with objectives, constraints, and designers. As mentioned in the noncooperative protocol in the previous section, the solution of a noncooperative design scenario is the possible intersection of the Rational Reaction Sets of every designer. For highly nonlinear problems, however, the exact equations of the RRS are seldom possible to obtain, and approximating functions have to be used. The most commonly used approximating functions are polynomial *response surface* equations (Koch *et al.* 1998). To be used, the design space of each designer has to be sampled: this is traditionally done using statistical experimentation (design of experiments—DOE) (Simpson *et al.* 1997). A conceptual outline for the construction of the RRS is illustrated in figure 1.

To approximate the RRS for Player 1, for example, the following procedure is carried out. First, different points are sampled in Players 2's non-linear design space (composed of n design variables $(x_{21}, \ldots, x_{2n})$) according to a specific DOE protocol (e.g. central composite design, full factorial, partial factorial). At each of those specified points, Player 1's model is solved. Once these steps are completed, a second-order polynomial (in our case) is fitted to relate $\mathbf{x}_2$ to $\mathbf{x}_1$.

Those response surface equations embody the strategies of the various designers, and their possible intersection represents the different equilibria of the design space.

Once again, these equilibria, known as Nash solution points, are individually stable but not necessarily collectively stable, and it is the aim of this paper to introduce methodologies to investigate these properties. The next section presents several methods, derived from nonlinear control theory, that can be used to determine the stability of equilibria.

4. Stability of equilibria

The focus of control theory is the analysis and the design of control systems. The analysis part involves the determination of the characteristics of a given system's behaviour, one of which is its equilibrium points and its stability. Techniques from this field can therefore be applied to the study of equilibria in engineering design, after making some changes to adapt them to this kind of problem. Since the focus of this paper is highly nonlinear design problems, the techniques that are used and described in this section are derived from *nonlinear* control theory (Slotine and Li 1991).

The previous section described how to obtain approximations for the RRS of each player involved in the design process. In a sequential approach to design, information is exchanged back and forth between the designers before reaching a final agreement. This can be compared to a *discrete time* control problem, similar to the time-series formulation presented in Chanron and Lewis (2003). The most general discrete time update equation is shown in equation (5):

$$\mathbf{x}(k+1) = f(\mathbf{x}(k)), \tag{5}$$

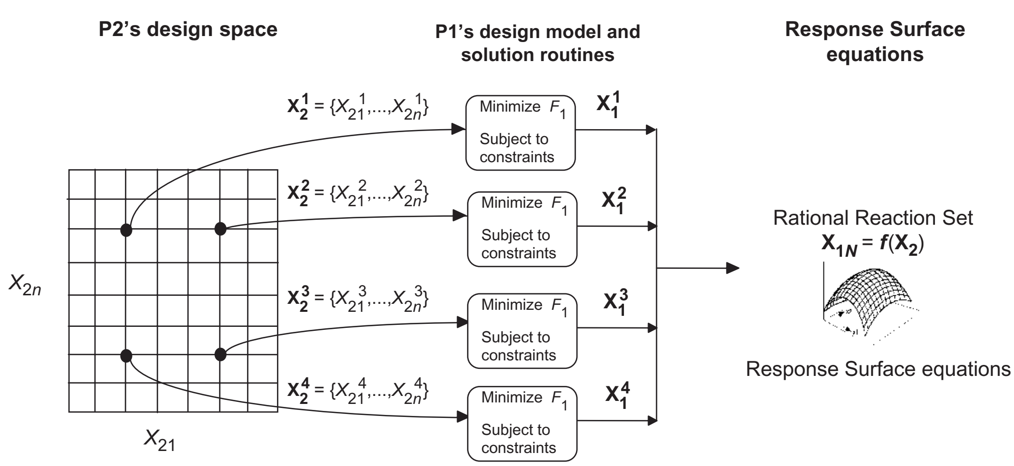

Figure 1. Construction of Rational Reaction Sets.

where f is a non-linear function of the states at the previous time steps, and $\mathbf{x}$ is the *state vector* of all the design variables of every designer. The function f is defined easily from the equations of the Response Surface approximations found using the method described in figure 1. f is an *autonomous* function, since it does not depend explicitly on time. Equation (5) is the key equation that will be used with every technique presented next.

4.1. *Finding the equilibrium points*

The intersection of the response surfaces approximating the RRS of every designer is an equilibrium point of the design space. This corresponds to a point where the control system studied has reached a *steady state*. Mathematically, one can solve for these steady-state points by setting $\mathbf{x}(k+1) = \mathbf{x}(k)$ in equation (5). Solving this equation gives us the set of equilibrium points. This set can be empty (there is therefore no Nash solution), or can have one or several equilibria, whose stability needs to be established. First, every equilibrium studied has to be moved to the origin by a simple change of variable in the update equation (5), since the methods derived from nonlinear control theory study the stability of the *origin*. By making this shift, the following techniques can be applied to the new update equation to study its stability. However, we first need to define *stability*, as used in nonlinear control theory. The definition of stability in the sense of *Lyapunov* is given next.

Stability: *The equilibrium state* $\mathbf{x} = 0$ *is said to be stable if, for any* $R > 0$, *there exists* $r > 0$, *such that if* $\|\mathbf{x}(0)\| < r$, *then* $\|\mathbf{x}(k)\| < R$ *for all* $k \geq 0$. *Otherwise, the equilibrium point is unstable.*

Lyapunov stability is definitely an interesting property for equilibria in decentralized design problems. However, other relevant information in the study of the collective stability of the Nash equilibria would be the properties of the region around the equilibrium. In other words, valuable information would be the *domain of attraction* of the equilibrium, which is the set of all points such that trajectories initiated at these points eventually converge to the origin, the equilibrium point (a trajectory represents the path that a design variable is taking in the design space).

This also introduces the concept of *asymptotic stability* which requires stability and the convergence of all the states to zero, collectively representing the origin. This is a stronger statement than the Lyapunov stability, since it requires stability *and* convergence. This distinction is significant, since the focus of this work is the convergence of the design variables in decentralized design problems. Using these definitions, the stability of the equilibria found can now be studied using the methods presented next.

4.2. *Linearization and local stability*

Lyapunov's linearization method is concerned with the *local* stability of the equilibrium of a nonlinear system. It is the formalization of the observation that a nonlinear system should behave similarly to its linearized approximation in a small range around the equilibrium. To do so, only the linear terms of the update equation (5) are retained. Equation (6) shows the linearization of the original non-linear system at the equilibrium point 0.

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k), \qquad \text{where} \qquad \mathbf{A} = \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}=0}. \quad (6)$$

This equation is similar to the update equation of a linear system, for which the study of stability is known. The stability of equilibria for linear decentralized design problems can be found in Chanron and Lewis (2003, 2004). The stability depends on the position of the eigenvalues of the matrix $\mathbf{A}$ with respect to the unit circle (Ogata 1995):

- The system is said to be *stable* if all the eigenvalues of $\mathbf{A}$ lie within the unit circle. Any eigenvalue outside the unit circle makes the system unstable.
- The system is said to be *marginally stable* if a simple eigenvalue of $\mathbf{A}$ lies on the unit circle. Also, the system becomes marginally stable if a single pair of conjugate complex eigenvalues lies on the unit circle.

We recommend the linearization to be the first step to check local convergence properties around the equilibrium. Indeed, the stability of the linearized system gives us interesting insights into the initial nonlinear system using the theorem of the *Lyapunov's linearization method* (Slotine and Li 1991).

- If the linearized system is strictly stable, then the origin is asymptotically stable for the actual nonlinear system.
- If the linearized system is unstable, then the origin is unstable (for the non-linear system).
- If the linearized system is marginally stable, one cannot conclude anything about the stability of the equilibrium point, since high-order terms affect the stability.

However, linearization gives only local information, and if more global information is required, other methods have to be used, and these are presented next.

4.3. *Lyapunov's direct method*

The Lyapunov's direct method allows conclusions to be drawn about the stability without using the complex stability definitions introduced earlier. This gives *necessary* conditions for the equilibrium to be stable and is based on the existence of a *Lyapunov function* (Franklin *et al*. 1998).

Lyapunov function: *If, in a ball* $\mathbf{B}_{R_0}$, *the scalar function* $V(k)$ *is positive definite and the function* $V(k+1) - V(k)$ *is negative semi-definite, then* $V(k)$ *is said to be a Lyapunov function for the system.*

With this definition of Lyapunov functions, the main Lyapunov theorem for stability can be introduced.

Lyapunov theorem: *If there exists a Lyapunov function, then the origin is stable. If, actually, the function* $V(k+1) - V(k)$ *is negative definite in* $\mathbf{B}_{R_0}$, *then the stability is asymptotic.*

This theorem can be applied directly once the update equation (5) is known. The difficulty lies in the process of searching for the Lyapunov function and in the fact that the Lyapunov theorem is a *sufficiency* theorem. If, for a particular choice of Lyapunov function candidate V, the conditions on $V(k+1) - V(k)$ are not met, one cannot draw any conclusions on the stability or instability of the system—the only conclusion one should draw is that a different Lyapunov function candidate or a different method should be tried to determine stability. In some situations, the graphical method can be used, and this is presented next.

4.4. *Graphical method*

In some cases, the interdependencies of the subsystems of the decentralized system offer interesting properties about the solutions of these problems. One of those cases occurs when a small number of disciplines are closely related to one another and independent from the rest of the design. Equation (7) shows an example for two interrelated disciplines and with quadratic update equations,

$$\begin{cases} \mathbf{x}(k+1) = a\mathbf{y}(k) + b\mathbf{y}^2(k) \\ \mathbf{y}(k+1) = c\mathbf{x}(k) + d\mathbf{x}^2(k). \end{cases} \tag{7}$$

For those coupled equations, an independent update equation can be found as $\mathbf{x}(k+1) = f(\mathbf{x}(k-1))$, or, written out:

$$\mathbf{x}(k+1) = ac\mathbf{x}(k-1) + \left[ad + bc^2\right]\mathbf{x}^2(k-1) + 2bcd\mathbf{x}^3(k-1) + bd^2\mathbf{x}^4(k-1). \tag{8}$$

This equation can be plotted: $\mathbf{x}(k+1)$ as a function of $\mathbf{x}(k-1)$, and the stability can be inferred from the position of this curve with respect to the quadrants and to the lines $\mathbf{x}(k+1) = \pm\mathbf{x}(k-1)$. This is the main idea behind the graphical method, which allows the visualization of the attraction basin. Also called the domain of attraction, the attraction basin is the set of points such that trajectories initiated at these points eventually converge to the origin (Slotine and Li 1991). Three main scenarios can occur:

- **Case 1:** The curve lies in the first and third quadrants. In this case, there is no sign change from one iteration to the next, since the update curve lies in the first and third quadrants: the image of a positive value design point will be a positive value design point; the image of a negative one will be another negative one. Therefore, the stability condition is that the curve has to stay within the lines of slope $+1$ and -1 of the design space ($y = \pm x$) so as to converge progressively towards the origin. This is illustrated using an example in figure 2.
- **Case 2:** The curve lies in the first and second quadrants (or, symmetrically, in the third and fourth). In this case, most of the design points will be positive. Indeed, if, at one iteration, a design point is negative, its image at the next iteration will be positive (since the update curve lies in the second quadrant). Besides, once a design point is positive, it will always remain positive in the future iterations as the update curve is in the first quadrant. Therefore, the upper limit of stability is when the update curve goes over the line $y = x$ (for similar reasons as in case 1), and the lower limit is found by finding the antecedent of this intersection in the second quadrant. This is illustrated using an example in figure 3.
- **Case 3:** The curve lies in the second and fourth quadrants. In this case, the sign of the current design point is constantly changing from one iteration to the next due to the position of the update in the quadrants: a negative design point will have a positive image, and vice versa. The stability conditions are now dynamic, as the upper limit of stability depends also on the relative position of the curve in the negative side of the design space, and similarly for the lower limit. Graphically, this is solved by plotting the symmetric image of the update curve with respect to $y = x$. This new curve is represented by a thick dash-dot line in the example of figure 4. The intersections of this curve with the initial update curve represent the lower and upper limit of stability.

These are the three main cases of the graphical method. A combination of these cases can also occur, but the

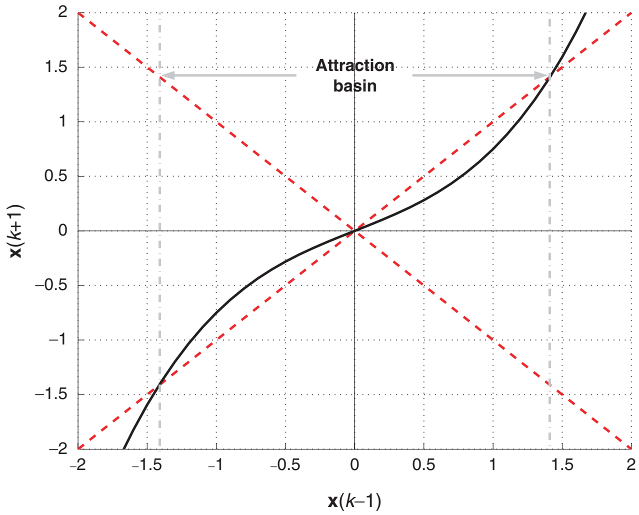

Figure 2. Solving using the graphical method—first case.

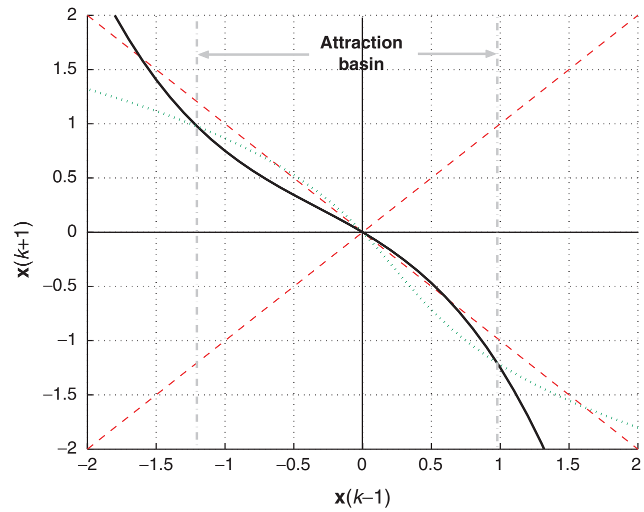

Figure 4. Solving using the graphical method—third case.

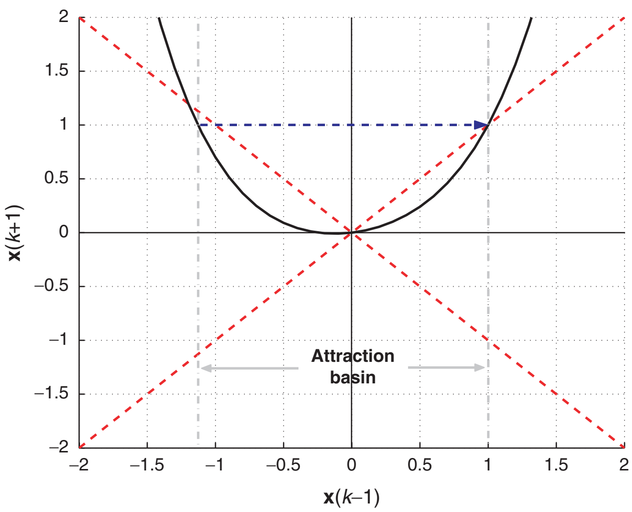

Figure 3. Solving using the graphical method—second case.

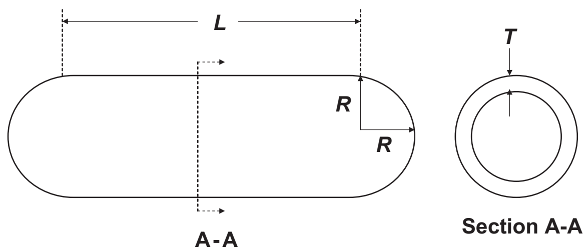

Figure 5. Thin-walled pressure vessel.

stability can always be found by applying the conditions stated here.

This section presented the main methods adapted from nonlinear control theory that can be used in solving the stability of equilibria in decentralized design. The next section introduces a case study on which these methods are tested.

5. Design of a pressure vessel

In this section, the design of a thin-walled pressure vessel with hemispherical ends, shown in figure 5, is used. The nomenclature for this case study taken from Lewis and Mistree (1998) is presented in table 2.

The vessel is to withstand a specified internal pressure P, and the material is also specified. There are two objectives: to minimize the weight and to maximize the volume of the cylinder, both subject to stress and geometry constraints. Although this is not naturally a multi-player problem, we consider in this paper that the design involves two design teams: (1) player VOL who wishes to maximize the volume and controls R and L and (2) player WGT who wishes to minimize the weight of the vessel and controls T. The objectives and constraints of the two players are shown in tables 3 and 4.

The specific data (problem constants) for this problem are as follows:

$$P = 3.89\,\text{klb}; \quad S_t = 35.0\,\text{klb}; \quad \rho = 0.283\,\text{lbs in}^{-3} \qquad (9)$$

As discussed in section 3, the equilibria of the design space are located at the intersection of the players' RRS, and we use Response Surface Methodology to approximate those RRS. Specifically, for the Volume player, values of the thickness, T, are sampled between 0.5 in

Table 2. Nomenclature of the pressure vessel.

W	Weight of the pressure vessel (lb)
V	Volume (in^3)
R	Radius (in)
T	Thickness (in)
L	Length (in)
P	Pressure inside the cylinder (klb)
S_t	Material allowable tensile strength (klb)
ρ	Density of the material (lbs in^{-3})
σ_{circ}	Circumferential stress (lbs in^{-2})

Table 3. Model of player VOL.

Maximize	$V(R,L) = (4/3)\pi R^3 + \pi R^2 L$
Design variables	R and L
Stress constraint	$\sigma_{\text{circ}} = (PR/T) \leq S_t$
Geometric constraints	$5T - R \leq 0$
	$R + T - 40 \leq 0$
	$L + 2R + 2T - 150 \leq 0$
Side constraints	$0.1 \leq R \leq 36$
	$0.1 \leq L \leq 140$

and 6.0 in and for each value of thickness, the Volume model is solved, resulting in values of the radius, R, and length, L. A second-order response surface is then developed using this set of input and output points to approximate R and L as a function of T. These resulting relationships are shown in equation (10) (Lewis and Mistree 1998). For the Weight player, values of R and L are sampled between 0.1 in and 36.0 in and 0.1 in and 140.0 in, respectively, using a Central Composite, Face-centred, Design (Myers and Montgomery 1995). For each value of R and L, the Weight model is solved for the thickness, T. A second-order response surface is also developed using this set of input and output points to approximate T as a function of R and L, as shown in equation (11) (Lewis and Mistree 1998).

$$\textbf{VOL} \quad \begin{aligned} R(T) &= 29.29 + 14.75T - 10.01T^2 \\ L(T) &= 85.45 - 34.45T + 20.10T^2 \end{aligned} \tag{10}$$

$$\textbf{WGT} \quad T(R,L) = 2 + 1.75R + 0.2445R^2. \tag{11}$$

It is interesting to note that the decision of designer Weight is independent of the value of L (the coefficients of L and L^2 were $(3.15 \times 10^{-5}R - 2.267 \times 10^{-5})$ and (8.667×10^{-7}), respectively, and therefore were removed). The intersection of these Rational Reaction Sets give the Nash solution(s). There is only one intersection here, so there is only one equilibrium point, given in equation (12):

$$\begin{aligned} R^N &= 28.4\,\text{in} \\ L^N &= 87.5\,\text{in} \\ T^N &= 3.09\,\text{in} \end{aligned} \tag{12}$$

In the remainder of the paper, we use normalized values from -1 to 1 for the three design variables. Tables 3 and 4 give the range of each design variable, which allows normalization of equations (10) and (11), and is shown in equation (13):

$$\begin{aligned} R(k+1) &= 0.626 + 0.822T(k) - 0.558T(k)^2 \\ L(k+1) &= 0.220 - 0.492T(k) + 0.287T(k)^2 \\ T(k+1) &= -0.455 + 0.636R(k) + 0.089R(k)^2. \end{aligned} \tag{13}$$

As mentioned in section 4, in order to study the stability of this equilibrium, we first need to move it to the origin. This is done by substituting $(\mathbf{x} + \mathbf{x}_N)$ for $\mathbf{x}$, where $\mathbf{x} = [R, L, T]^T$ is the normalized state vector, and $\mathbf{x}_N$ is the normalized state vector evaluated at the equilibrium. The new update equation is shown in equation (14):

$$\begin{aligned} R(k+1) &= 0.887T(k) - 0.558T(k)^2 \\ L(k+1) &= -0.525T(k) + 0.287T(k)^2 \\ T(k+1) &= 0.739R(k) + 0.089R(k)^2. \end{aligned} \tag{14}$$

Shifting the Nash equilibrium to the origin creates a change in the bounds of the design variables. In the new design space, R is in the interval $[-1.5766; 0.4234]$, T in $[-0.9418; 1.0582]$, and L in $[-1.2495; 0.7505]$.

Table 4. Model of player WGT.

Minimize	$W(R,T,L) = \rho\big[(4/3)\pi(R+T)^3 + \pi(R+T)^2 L - \big((4/3)\pi R^3 + \pi R^2 L\big)\big]$
Design Variables	T
Stress constraint	$\sigma_{\text{circ}} = (PR/T) \leq S_t$
Geometric constraints	$5T - R \leq 0$
	$R + T - 40 \leq 0$
	$L + 2R + 2T - 150 \leq 0$
Side constraints	$0.5 \leq T \leq 6$

5.1. *Local stability*

The first method to investigate the stability in the neighbourhood of the equilibrium is *linearization*. In this case, it is straightforward, as we just need to keep the linear terms of equation (14), and put them into a matrix format, as shown in equation (6), to find the state matrix **A**,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0.887 \\ 0 & 0 & -0.525 \\ 0.739 & 0 & 0 \end{bmatrix}. \tag{15}$$

A simple check of the eigenvalues of **A** gives us the stability properties of the linearized system,

$$\text{Eigenvalues of } \mathbf{A} = \{0,\ 0.8096,\ -0.8096\}. \tag{16}$$

The stability is determined by the value of the spectral radius (maximum absolute value of the eigenvalues of a matrix), which here is: $r_\sigma(\mathbf{A}) = 0.8096$. Since it is strictly less than 1, the linearized system is strictly stable, and therefore the initial non-linear system is *asymptotically stable* (Chanron and Lewis 2004).

5.2. *Lyapunov stability*

The Lyapunov direct method, as presented earlier, gives a simple way to address the issue of stability but does not provide any coherent methodology for constructing the Lyapunov function. This is the purpose of the SOSTOOLS software (Papachristodoulou and Prajna 2002), which uses SeDuMi (Sturm 1999) as the semi-definite programming solver, and which we use in this section to find a Lyapunov function.

Equation (14) shows that the design variables R and T are closely related to each other, while design variable L is only a function of T. Equation (17) is therefore used as the new update equation to demonstrate the stability of the entire system,

$$\begin{aligned} R(k+1) &= 0.887T(k) - 0.558T(k)^2 \\ T(k+1) &= 0.739R(k) + 0.089R(k)^2. \end{aligned} \tag{17}$$

SOSTOOLS allows for an algorithmic construction of Lyapunov functions. We assume the following form for the candidate Lyapunov function, and use SOSTOOLS to solve for the coefficients while ensuring the positive definiteness of V, and the negative definiteness of $V(k+1) - V(k)$,

$$V = aR^2 + bRT + cT^2. \tag{18}$$

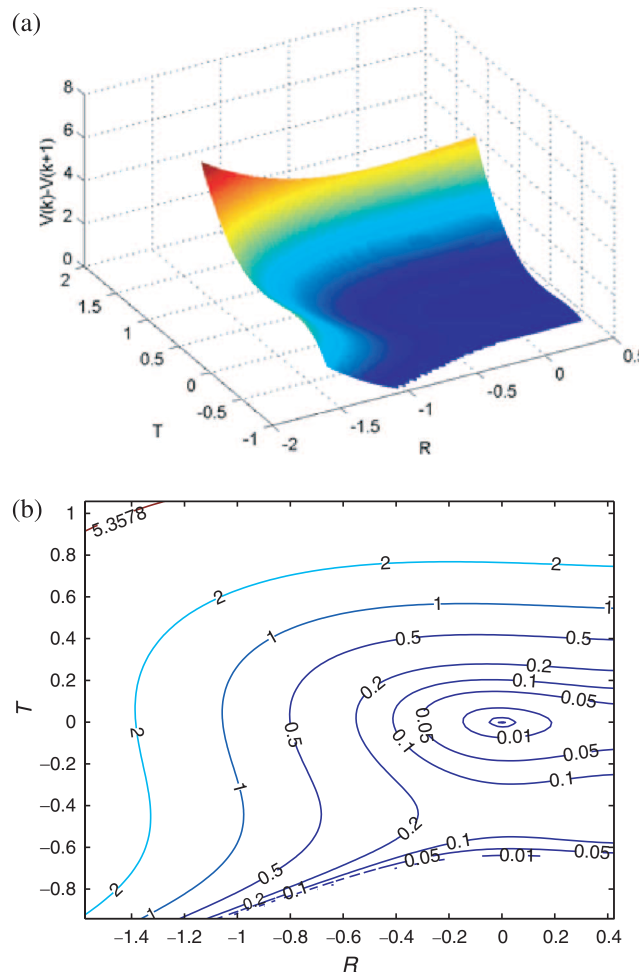

Figure 6. Positive definiteness of $V(k) - V(k+1)$. (a) Surface. (b) Contour.

The solution given by SOSTOOLS is shown in equation (19) and put into a matrix form to prove its positive definiteness.

$$V = 2.54R^2 + 0.40RT + 3.96T^2 \tag{19}$$

$$V = [R \quad T] \cdot \begin{bmatrix} 2.54 & 0.2 \\ 0.2 & 3.96 \end{bmatrix} \cdot \begin{bmatrix} R \\ T \end{bmatrix}. \tag{20}$$

The eigenvalues of the matrix are strictly positive (2.5165 and 3.9870), thus proving the positive definiteness of V.

The second necessary condition for V to be a Lyapunov function is that $V(k+1) - V(k)$ has to be negative definite. In the form proposed in equation (18), negative definiteness cannot be found for R and T in the entire design space. However, figure 6 shows that only a small portion of the design space (for small values of T and large values of R) does not guarantee stability, given the Lyapunov function proposed here.

5.3. *Graphical method*

We now use the graphical method on the same case study to demonstrate its applicability. Starting from equation (17), an update equation linking $T(k+1)$ to $T(k-1)$ can be found. This is shown in equation (21) and plotted in the $(T(k-1), T(k+1))$ plane:

$$T(k+1) = 0.028T(k-1)^4 - 0.088T(k-1)^3 - 0.342T(k-1)^2 + 0.655T(k-1). \quad (21)$$

This graph is shown in figure 7, along with the resulting quadrants. Studying the stability of this update equation will determine the stability for the entire problem, since equation (21) was obtained using the set of coupled equations shown in equation (17). Indeed, if T converges to its equilibrium value, then R and L will also converge based on equation (14).

Effective insight on the stability can be gained by studying this figure. Indeed, the update curve is in the first and third quadrants *only*. We can therefore use the results found in the first case of section 4.4. The update curve stays clearly within the lines $T(k+1) = \pm T(k-1)$ for the entire design space. Therefore, the design process is stable around the origin, and all trajectories that start in this design space will end at the origin.

Figure 7 also shows two trajectories for two initial conditions ($T_0 = 0.4$ and $T_0 = -0.8$). It shows the first iterations of the design process, and the convergence towards the origin of the design space. The origin of the space in this figure actually represents the Nash equilibrium shown in equation (12). Therefore, the attraction basin of this Nash equilibrium is the entire design space. Note that it is not necessary for the entire design space to be in the attraction basin, even if there is a unique equilibrium, as some initial conditions might create a divergent pattern as noted in Vincent (1983) and Chanron and Lewis (2003). Also, the fact that approximations were used to find the equations of the response surfaces might affect the quality of the stability prediction. However, these results are consistent with the stability results presented in Lewis and Mistree (1998), thus validating the methods used in this paper.

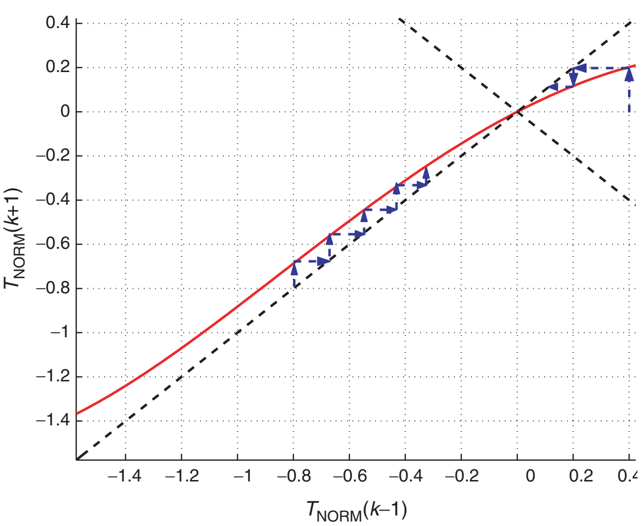

Figure 7. Graphical solution of the case study.

6. Conclusion

Most engineering systems are multidisciplinary in nature and therefore require knowledge from several design teams. This, along with other constraints, forces the decentralization of decisions. Therefore, the decision-makers involved in these kinds of design processes need to understand the dynamics involved in order to find a final optimal design. This paper develops a theoretical approach to modelling these dynamics and, specifically, the stability of the equilibrium points of the decentralized problem. It extends the existing results about stability to more complex problems with highly non-linear objectives and constraints. This is done by applying concepts adapted from nonlinear control theory to the study of equilibrium solutions in engineering design. A case study presents the first direct applications of these methods on a design example.

By applying the concepts in this paper, design engineers could avoid a great deal of iteration and wasted resources by determining the convergence and stability characteristics of their design process. However, the broader goal of this research is for design engineers to understand the complications that decentralization creates, including the possibility of creating a divergent system. The research strives to give them a broader vision of the issues created by the decentralization of decisions and help them make better decisions (e.g. decisions to avoid divergent processes). This work should also help the research community to propose new methods to assist engineers in distributed design environments. This is part of future work that includes applying the concepts presented in this paper to other case studies, generalizing the results to higher-order approximations, and drawing conclusions that will be used as a foundation to a comprehensive theory for decentralized processes.

Acknowledgements

We would first like to acknowledge Stephen Prajna for his help using the SOSTOOLS software to generate the Lyapunov function. We would also like to thank the National Science Foundation, grants DMII-9875706 (for the preliminary convergence work that led to this paper) and DMII-0322783 (for the stability work of this paper) for their support of this research.

References

B. Allen, "Game theoretic models of search in multi-agent process planning systems", in *Special Interest Group on Manufacturing (SIGMAN) Workshop, International Joint Conferences on Artificial Intelligence*, 2001.

R.J. Balling and J. Sobieczcanski-Sobieski, "Optimization of coupled systems: a critical overview of approaches", in *5th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, AIAA-94-4330-CP, 1994.

R. Braun, P. Gage, I. Kroo and I. Sobieski, "Implementation and performance issues in collaborative optimization", in *6th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, AIAA 96-4017, 1996.

V. Chanron and K. Lewis, "A study of convergence in decentralized design", in *ASME Design Engineering Technical Conference*, DETC03/DAC-48782, 2003.

V. Chanron and K. Lewis, "Convergence and stability in distributed design of large systems", in *ASME Design Engineering Technical Conferences*, DETC2004-57344, 2004.

L. Chen and S. Li, "Concurrent parametric design using a multifunctional team approach", in *ASME Design Engineering Technical Conferences*, DETC 2001/DAC-21038, 2001.

A. Fink and T. Singh, "Discrete sliding mode controller for pressure control with an electrohydraulic servovalve", in *IEEE Conference on Control Applications* (*CCA*), 1998, pp. 378–382 .

G.F. Franklin, J.D. Powell and M.L. Workman, *Digital Control of Dynamic Systems*, 3rd ed., Reading, Massachusetts: Addison-Wesley, 1998.

C. Guarino Lo Bianco and A. Piazzi, "A hybrid algorithm for infinitely constrained optimization", *Int. J. Syst. Sci.*, 32(1), pp. 91–102, 2001.

G. Hernandez, C.C. Seepersad and F. Mistree, "Designing for maintenance: a game theoretic approach", *Eng. Optim.*, 34(6), pp. 561–577, 2002.

P.N. Koch, D. Mavris and F. Mistree, "Multilevel, partitioned response surfaces for modeling complex systems", in *7th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, AIAA-98-4958, 1998.

R. Krishnamachari and P. Papalambros, "Hierarchical decomposition synthesis in optimal systems design", *J. Mech. Design*, 119(4), pp. 448–457, 1997.

A. Kusiak and J. Wang, "Decomposition of the design process", *J. Mech. Design*, 115(4), pp. 687–695, 1993.

H. Lee and S. Whang, "Decentralized multi-echelon supply chains: incentives and information", *Manage. Sci.*, 45(5), pp. 633–640, 1999.

T.S. Lee and S. Ghosh, "On the concept of "stability" in asynchronous distributed decision-making systems", *IEICE Trans. Commun.*, E83-B(5), pp. 1023–1038, 2000.

K. Lewis and F. Mistree, "Modeling the interactions in multidisciplinary design: A game theoretic approach", *AIAA J. Aircraft*, 35(8), pp. 1387–1392, 1997.

K. Lewis and F. Mistree, "Collaborative, sequential and isolated decisions in design", *J. Mech. Design*, 120, pp. 643–652, 1998.

C. Loch, J. Mihm and A. Huchzermeier, "Concurrent engineering and design oscillations in complex engineering projects", *Concurr. Eng.*, 11(3), pp. 187–199, 2003.

M. Marston and F. Mistree, "Game-based design: a game theoretic extension to decision-based design", in *ASME Design Engineering Technical Conferences*, DETC2000/DTM-14578, 2000.

C. McCulley and C.L. Bloebaum, "A genetic tool for optimal design sequencing in complex engineering systems", *Struct. Optim.*, 12, pp. 186–201, 1996.

N. Michelena, H. Park and P. Papalambros, "Convergence properties of analytical target cascading", in *9th AIAA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, AIAA 2002-5506, 2002.

R.H. Myers and D.C. Montgomery, *Response Surface Methodology*, New York: Wiley, 1995.

K. Ogata, *Discrete-Time Control Systems*, 2nd ed., Upper Saddle River, NJ: Prentice-Hall, 1995.

A. Papachristodoulou and S. Prajna, "On the construction of Lyapunov functions using the sum of squares decomposition", in *IEEE Conference on Decision and Control*, 2002.

V. Pareto, *Manuale di Economica Polittica*, Milan: Societa Editrice Libraia, 1906. Translated into English by A.S. Shwier, as *Manual of Political Economy*, New York: Macmillan, 1971.

E. Prewitt, "Fast-cycle decision making", *Harv. Manage. Update*, 3(2), pp. 8–9, 1998.

J.J.R. Rao, K. Badhrinath, R. Pakala and F. Mistree, "A study of optimal design under conflict using models of multi-player games", *Eng. Optim.*, 28, pp. 63–94, 1997.

T.W. Simpson, J. Peplinski, P.N. Koch and J.K. Allen, "On the use of statistics in design and the implications for deterministic computer experiments", in *ASME Design Engineering Technical Conferences*, DETC97/DTM-3881, 1997.

J.-J.E. Slotine and W. Li, *Applied Nonlinear Control*, Englewood Cliffs, NJ: Prentice-Hall, 1991.

J. Sobieszczanski-Sobieski, B.J. James and A.R. Dovi, "Structural optimization by multilevel decomposition", *AIAA J.*, 23(11), pp. 1775–1782, 1985.

J.F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones", *Optim. Meth. Softw.*, 11–12, pp. 625–653, 1999.

T.L. Vincent, "Game theory as a design tool", *J. Mechanisms, Transmissions Autom. Design*, 105, pp. 165–170, 1983.

R.I. Whitfield, A.H.B. Duffy, G. Coates and W. Hills, "Distributed design coordination", *Res. Eng. Design*, 13, pp. 243–252, 2002.

B. Xu, "Delay-independent stability criteria for linear continuous systems with time-varying delays", *Int. J. Syst. Sci.*, 33(7), pp. 543–550, 2002.

C. Xu, "Rational behaviour and cooperation degree in competitive situations", *Int. J. Syst. Sci.*, 30(4), pp. 369–377, 1999.

Vincent Chanron received a Master of Science in aerospace engineering from the Ecole Nationale Supérieure d'Ingénieurs de Constructions Aéronautiques (ENSICA) in Toulouse (France) in 2002. He also received a Master of Science in mechanical engineering from the University at Buffalo in 2002. He is completing his Ph.D. in August 2005 and his dissertation focuses on the dynamics and convergence of decentralized design processes. Starting in September 2005, he will be working for Airbus in Hamburg, Germany on the research and development of vertical tail aircraft.

Tarunraj Singh received his B.E., M.E and Ph.D. degrees in mechanical engineering from Bangalore University, Indian Institute of Science and the University of Waterloo respectively. He was a post doctoral fellow in the Aerospace Engineering Dept. of Texas A & M University prior to starting his tenure at the University at Buffalo in 1993, where he is currently a Professor in the Department of Mechanical and Aerospace Engineering. He was a von Humboldt fellow and spent his sabbatical at the Technische Universität Darmstadt in Germany and at the IBM Almaden Research center in 2000–2001. He was a NASA Summer Faculty Fellow at the Goddard Space Flight Center in 2003. His research interests are in robust vibration control, optimal control, nonlinear estimation and intelligent transportation. His research is supported by the National Science Foundation, AFOSR, NSA, Office of Naval Research and various industries. He has published over 100 refereed journal and conference papers and has over 30 invited seminars at various universities and research laboratories.

Kemper Lewis received his B.A. in mathematics and B.S. in mechanical engineering from Duke University, and his M.S. and Ph.D. degrees in mechanical engineering from Georgia Institute of Technology. His tenure at the University at Buffalo began in 1996, where he is currently an Associate Professor in the Department of Mechanical and Aerospace Engineering. He is also Executive Director of the New York State Center for Engineering Design and Industrial Innovation (NYSCEDII). His research areas include design theory, distributed design, reconfigurable systems, multidisciplinary optimization, and multiobjective design. He has over 100 technical publications in these research areas. His awards include the Society of Automotive Engineers Teetor Award, the ASME Black and Decker Best Paper Award, the State University of New York Chancellor's Award for Excellence in Teaching, and the National Science Foundation Career Award.