State Uncertainty Propagation in the Presence of Parametric Uncertainty and Additive White Noise

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The focus of this work is on the development of a framework permitting the unification of generalized polynomial chaos (gPC) with the linear moment propagation equations, to accurately characterize the state distribution for linear systems subject to initial condition uncertainty, Gaussian white noise excitation and parametric uncertainty which is not required to be Gaussian. For a fixed value of parameters, an ensemble of moment propagation equations characterize the distribution of the state vector resulting from Gaussian initial conditions and stochastic forcing, which is modeled as Gaussian white noise. These moment equations exploit the gPC approach to describe the propagation of a combination of uncertainties in model parameters, initial conditions and forcing terms. Sampling the uncertain parameters according to the gPC approach, and integrating via quadrature, the distribution for the state vector can be obtained. Similarly, for a fixed realization of the stochastic forcing process, the gPC approach provides an output distribution resulting from parametric uncertainty. This approach can be further combined with moment propagation equations to describe the propagation of the state distribution, which encapsulates uncertainties in model parameters, initial conditions and forcing terms. The proposed techniques are illustrated on two benchmark problems to demonstrate the techniques' potential in characterizing the non-Gaussian distribution of the state vector. [DOI: 10.1115/1.4004072]

1 Introduction

In applications from hurricane forecasting to robust control of cranes on ships transferring freight on the high seas, to predicting the probability of an asteroid path intersecting the path of the earth, one is confronted with the problem of dealing with approximate models and unmodeled disturbances. The benefits accruing from efficient deployment of assets and evacuation of populations in the case of a hurricane landfall are tremendous, if the landfall uncertainty has been accurately characterized. These examples lucidly illustrate the need for algorithms that can accurately forecast dynamic system states and characterize the associated uncertainties. The importance of quantifying the effects of uncertainties is reflected by the formulation of challenge problems by Oberkampf et al. [1] for an algebraic problem and an ordinary differential equation representing the dynamics of a spring–mass–dashpot.

The mathematical models used to represent physical processes often reflect the many assumptions and simplifications required to permit determination of a tractable model. The solution state $\mathbf{x}$ of these models is therefore uncertain and may be described by a time-dependent probability density function (pdf) $p(t, \mathbf{x})$. The uncertainty inherent in these models is either due to a lack of complete description of the system, i.e., model truncation, or due to the uncertainty in model parameters and input to the system. Such models may be characterized by uncertain model parameters and stochastic forcing terms. Together these factors cause overall accuracy to degrade as the state evolves. The uncertainty in the solution of the models could also arise due to the uncertainty in initial and boundary conditions driving the models. Robust modeling of the propagation of these uncertainties is important to accurately quantify the uncertainty in the solution at any future time. A naive approach to account for all uncertainties is to sample all possibilities using the model and have some mechanism for averaging the outputs appropriately. Unfortunately for many classes of realistic models, the computational cost of this approach makes it infeasible.

Uncertainty propagation in various kinds of dynamical systems and physical processes has been studied extensively in various fields of engineering, finance, physical, and environmental sciences [2–6]. The exact propagation of the state pdf for linear dynamical systems subject to initial condition and temporal stochastic disturbance generally modeled as a white noise process is given by a finite number of moment propagation equations [7,8]. On the other hand, methods based on generalized polynomial chaos (gPC) have emerged as powerful tools to propagate time-invariant parametric uncertainty through an otherwise deterministic system of equations, to predict a distribution of outputs [9–14]. Nagy and Bratz [13] use power series expansion and polynomial chaos expansion to quantify the uncertainty in the output of nonlinear systems, and illustrate it on a batch crystallization process. Hover and Triantafyllou [12] use Polynomial Chaos to analysis the transient response and stability of nonlinear systems due to uncertainties in model parameters. Fisher and Bhattacharya [14] provide a framework for designing infinite time horizon Linear Quadratic Regulator (LQR) controllers in the presence of probabilistic uncertainties in the model parameters, and Singh et al. [15] uses the gPC method to design robust input shapers for precise control of mechanical systems. While gPC can efficiently characterize the state uncertainty due to time-invariant random parameters having arbitrary probability distributions, using gPC series expansion for the time-varying stochastic forcing terms is computationally expensive. For stationary stochastic processes, which are correlated in time, a smaller set of terms in the expansion may be enough to model the random processes to keep the computation feasible [11,16]. However, for a model driven by white noise, an infinite number of terms is required to model the process [11]. The computational costs increase exponentially with the increase in the number of time steps, due to the increase in the stochastic dimensionality even for a linear dynamical system.

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Contributed by the Dynamic Systems Division of ASME for publication in the Journal of Dynamic Systems, Measurement, and Control. Manuscript received October 12, 2009; final manuscript received March 14, 2011; published online August 5, 2011. Assoc. Editor: Swaroop Darbha.

While these methods work very well and even provide exact description of the uncertainty propagation for linear dynamical systems subject to either initial condition and temporal stochastic disturbance modeled as white noise process or time-invariant parametric uncertainty, the main challenge lies in characterizing the uncertainty in the system states due to *both parametric and temporal stochastic uncertainties* simultaneously. A multiple-model estimation method involving a bank of Kalman filters with prior probabilities assigned to each filter has been proposed in Ref. [17] to manage both uncertainties. The main limitation of this method is that it assumes the uncertain parameters belong to a discrete set. The uncertain parameter vector is quantized to a finite number of grid points with known prior probabilities. The state conditional mean and covariance are propagated for each model corresponding to a grid point, and the first two moments of system states are computed by a weighted average of the moments corresponding to various prior models. Alternatively, the uncertain parameters are appended to the system state vector to characterize the effect of their uncertainty, which results in a nonlinear dynamical system even in the case that the original system dynamics were linear. Several approximate techniques may be used to study the uncertainty propagation problem through the resulting nonlinear dynamical systems, the most popular being Monte Carlo (MC) methods [18], Gaussian closure [19], equivalent linearization [20], and stochastic averaging [21,22]. Except for the Monte Carlo approach, most of the methods are similar in several respects, incorporating linear approximations to the nonlinear system response, or involving the propagation of only a few moments (often, just the mean and the covariance) of the pdf. These methods have been shown to work well if the amount of uncertainty is small and there is adequate local linearity. Another class of methods, often used with models involving many uncertain parameters, are the various sampling strategies [23]. The uncertainty distributions are taken into account by appropriately sampling values from known or approximated prior distributions, and the model is run repeatedly for those values to obtain a distribution of the outputs and estimate the posterior pdf. Monte Carlo or other sampling based methods require extensive computational resources and effort, and become increasingly infeasible for high-dimensional dynamic systems [24].

In short, although many algorithms exist in the literature to accurately characterize the uncertainty propagation problem for linear and nonlinear dynamical systems, none of them is able to incorporate *both parametric and temporal stochastic uncertainties* simultaneously with scalable computational costs, even for linear dynamical systems. In this work, we focus on developing analytical means to accurately characterize the state pdf of a linear system subject to initial condition uncertainty, white noise excitation, and possibly non-Gaussian parametric uncertainty. The Bayesian framework is used to characterize the effect of uncertainty due to stochastic forcing, while the gPC framework is used to characterize the effect of parametric uncertainty.

The remainder of this paper is organized as follows: first the theory which integrates the Bayesian framework with the gPC framework to characterize the uncertainty in parameters, initial conditions and due to white noise excitation is presented, followed by the details on the proposed approaches. The proposed methods are illustrated on benchmark examples, and the results are compared with the Monte Carlo solutions. Finally, the conclusions and directions for future work are presented.

2 Hybrid Bayesian-gPC Uncertainty Propagation

In conventional deterministic systems, the system state assumes a fixed value at any given instant of time. However, in stochastic dynamics, it is a random variable and for linear time-invariant systems driven by Gaussian white noise, its time evolution is given by the following stochastic differential equation:

$$\dot{\mathbf{x}} = \mathbf{A}(\Theta)\mathbf{x} + \mathbf{B}(\Theta)\mathbf{u} + \mathbf{G}(\Theta)\eta,\ \mathbf{x}(t_0) = \mu_0 \tag{1}$$

where, $\mathbf{x}(t) \in \mathbb{R}^n$ represents the stochastic system state vector at time t, $\Theta \in \mathbb{R}^m$ the uncertain time-invariant system parameters, $\mathbf{u}$ the deterministic forcing terms, $\eta \in \mathcal{N}(t, \omega; 0, Q)$ the stochastic forcing zero mean Gaussian white noise process with the correlation function $\mathbf{Q}\delta(t_1 - t_2)$, and μ_0 the random initial state modeled as Gaussian. The Gaussian white noise process η is assumed to be uncorrelated in time and with other uncertainties in model parameters and initial conditions. The uncertain parameters Θ are assumed to be functions of a random vector ξ having a known pdf $p(\xi)$, with common support Ω. The uncertainty associated with the state vector $\mathbf{x}$ is usually characterized by time parameterized state pdf $p(t, \mathbf{x}, \Theta)$. In essence, the study of stochastic systems reduces to finding the nature of time evolution of the initial system-state pdf $p(t_0, \mathbf{x}_0, \Theta)$ generally assumed to be Gaussian with mean μ_0 and covariance Σ_0. A key idea of this work is to *replace the time evolution of state in the dynamic model by the time evolution of state probability distribution* as shown in Fig. 1. By

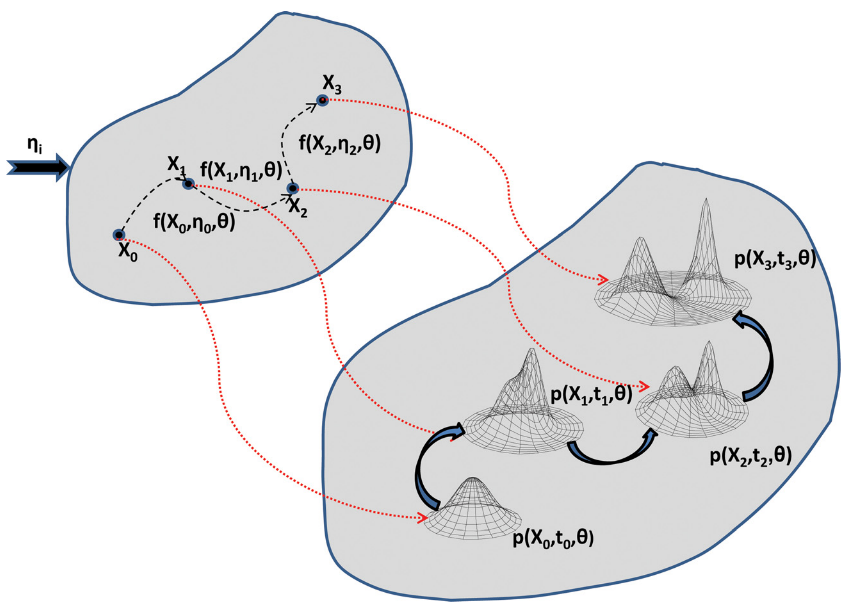


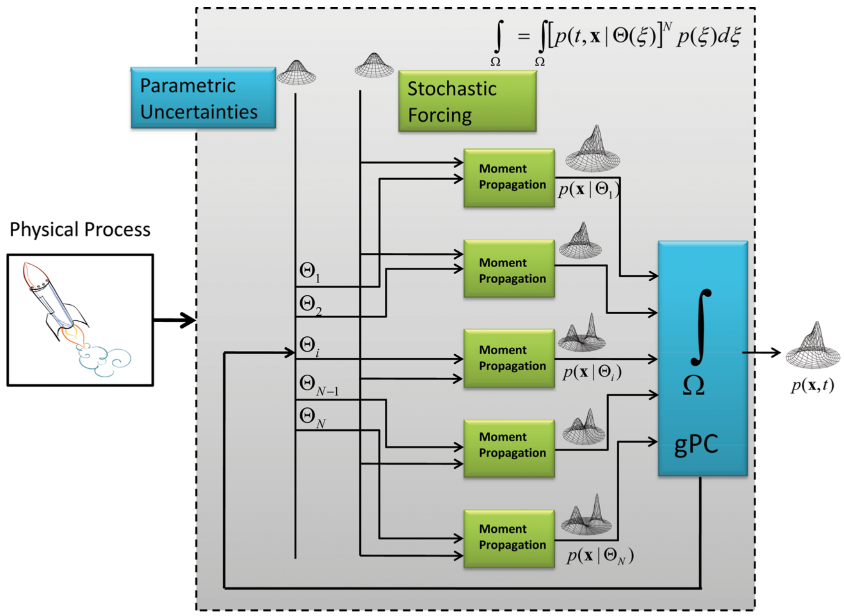
Fig. 1 State and pdf transition

computing the full probability density functions, we can monitor the evolution of uncertainty, represent multimodal distributions, incorporate complex prior models, and exploit Bayesian belief propagation, both through space and time.

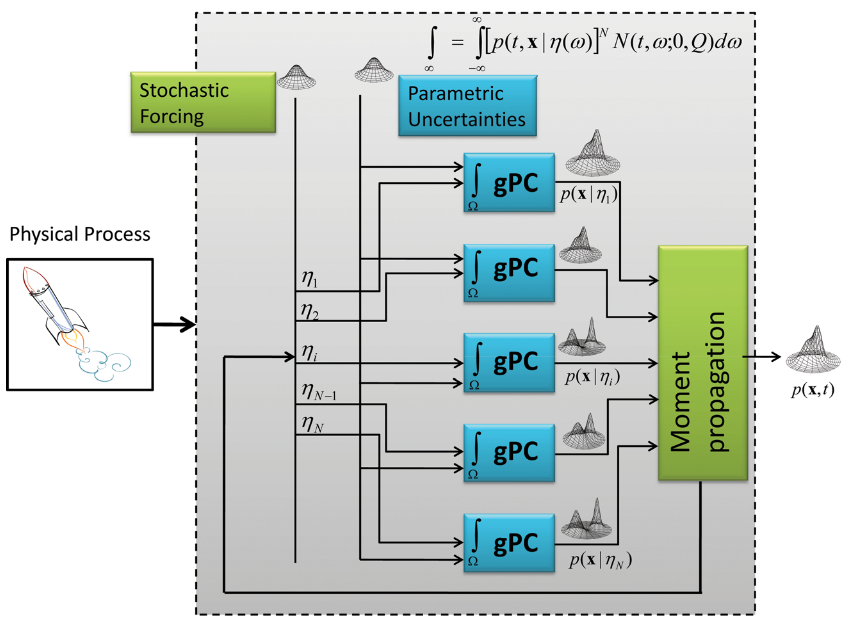
It is well-known that for any fixed value of Θ, the system state pdf of the linear model described by Eq. (1) is Gaussian. The propagation of uncertainty in the stochastic model of Eq. (1) for a given value of parameter Θ_i can be described using the state mean and covariance propagation when the forcing term is additive white Gaussian noise (AWGN) and initial conditions are Gaussian. Notice that if one appends the state vector $\mathbf{x}$ with unknown parameter vector Θ, the resulting state-space model will be nonlinear in nature and propagation of just mean and covariance would not suffice to accurately propagate the uncertainty. On the other hand, the uncertainty propagation of Eq. (1) can be efficiently described using polynomial chaos series expansion for a given realization η_i of the input. Since the stochastic forcing terms are independent random variables at different times, using gPC expansion to model the stochastic forcing terms is computationally intractable.

The main idea of this work is to *marry gPC with linear moment propagation equations*, to properly integrate the conditional distribution function and thence determine the posterior distribution of the full system. Owing to the independence of parametric and forcing uncertainties, the posterior state distribution can be obtained by conditioning first on either of these uncertainties as shown in Fig. 2.

For a fixed value of parameter $\Theta = \Theta_i$, an ensemble of moment propagation equations provides an output distribution owing to stochastic forcing. This approach can be further combined with



(*a*) Method 1: Conditioning first on uncertain parameters



(*b*) Method 2: Conditioning first on Gaussian stochastic forcing

Fig. 2 Proposed ideas for uncertainty propagation through a stochastic linear dynamical system: (*a*) method 1: conditioning first on uncertain parameters and (*b*) method 2: conditioning first on Gaussian stochastic forcing

polynomial chaos to describe the propagation of a combination of uncertainties in model parameters, initial conditions, and forcing terms. Varying the Θ inputs according to gPC approach, and integrating via quadrature, one can obtain the joint likelihood function as shown in Fig. 2(*a*). Similarly, for a fixed realization of the stochastic forcing process $\eta = \eta_i$, the gPC approach provides an output distribution owing to parametric uncertainty. This approach can be further combined with moment propagation equations to describe the propagation of a combination of uncertainties in model parameters, initial conditions, and forcing terms, as shown in Fig. 2(*b*).

3 Uncertainty Propagation

3.1 Method 1: Conditioning First on Uncertain Parameters. For a particular realization of the uncertain model parameters, the system states of the linear model described by Eq. (1) are Gaussian due to AWGN and Gaussian initial condition. The conditional state pdf $p(\mathbf{x}|\Theta)$ is a normal distribution with mean μ and covariance Σ, i.e., $p(\mathbf{x}|\Theta) = \mathcal{N}(\mathbf{x}; \mu, \Sigma)$. The two conditional moments of the model states $\mathbf{x}$ are given by the following equations:

$$\dot{\mu} = \mathbf{A}(\Theta)\mu + \mathbf{B}(\Theta)\mathbf{u} \tag{2}$$

$$\dot{\Sigma} = \mathbf{A}(\Theta)\Sigma + \Sigma\mathbf{A}^T(\Theta) + \mathbf{G}(\Theta)\mathbf{Q}\mathbf{G}^T(\Theta) \tag{3}$$

These conditional moment propagation equations are exact and depend only on the initial moments and the model parameters. These expressions can be used to obtain different realizations for μ and Σ, by Monte Carlo sampling of the random vector Θ. Each realization of μ and Σ represents the mean and covariance of the conditional distribution of $\mathbf{x}$ corresponding to a particular realization of the uncertain model parameters. As noted earlier, for a given realization of the model parameters, the distribution of $\mathbf{x}$ is Gaussian due to the additive white Gaussian forcing term. The complete distribution of the original state vector $\mathbf{x}$ at any time t, can thus be estimated from its realizations obtained by independent Monte Carlo sampling of the various Gaussian distributions resulting from the various samples of $\Theta(\xi)$ as

$$p(t,\mathbf{x}) = \int_{\Omega} p(t,\mathbf{x}|\Theta(\xi))p(\xi)d\xi \tag{4}$$

$$= \int_{\Omega} \mathcal{N}(t,\mathbf{x};\mu(t,\Theta),\Sigma(t,\Theta))p(\xi)d\xi \tag{5}$$

It is well-known that the Monte Carlo sampling requires extensive computational resources and effort, and become increasingly infeasible for high-dimensional dynamic systems [24]. To avoid Monte Carlo sampling, we make use of gPC to obtain the distribution for μ and covariance Σ in terms of the random variable ξ.

3.1.1 Polynomial Chaos. Polynomial chaos is a term originated by Norbert Wiener in 1938 [9] to describe the members of the span of Hermite polynomial functionals of a Gaussian process. According to the Cameron–Martin Theorem [25], the Fourier–Hermite polynomial chaos expansion converges, in the L^2 sense, to any arbitrary process with finite variance (which applies to most physical processes). This approach is combined with the finite element method to model uncertainty in Ref. [11]. This has been generalized in Ref. [10] to efficiently use the orthogonal polynomials from the Askey-scheme to model various probability distributions.

The basic goal of the approach is to approximate the stochastic system states in terms of a finite-dimensional series expansion in the infinite-dimensional stochastic space. The completeness of the space allows for the accurate representation of any random variable, with a given pdf, by a suitable projection on the space spanned by a carefully selected basis. The basis can be chosen for a given pdf, to represent the random variable with the fewest number of terms. For example, the Hermite polynomial basis can be used to represent random variables with Gaussian distribution using only two terms. For dynamical systems described by parameterized models, the unknown coefficients are determined by minimizing an appropriate norm of the residual.

Combining the state mean and covariance terms into a new state vector $\mathbf{z} \in \mathbb{R}^N$ where $N = n(n+3)/2$, the propagation equations represent an augmented model describing the evolution of $\mathbf{z}$ without a stochastic forcing term

$$\dot{\mathbf{z}} = \mathbf{L}(\Theta)\mathbf{z} + \mathbf{M}(\Theta)\mathbf{w} \tag{6}$$

where $\mathbf{L}(\Theta) \in \mathbb{R}^{N\times N}$ is the augmented system matrix, $\mathbf{M}(\Theta) \in \mathbb{R}^{N\times q}$ is the augmented input matrix corresponding to the transformed input vector $\mathbf{w} \in \mathbb{R}^q$ for the augmented model.

Applying gPC, each of the uncertain states and parameters can be expanded approximately by the finite-dimensional Wiener–Askey polynomial chaos [10] as

$$z_i(t,\xi) = \sum_{r=0}^{P} z_{ir}(t)\phi_r(\xi) = \mathbf{z}_i^T(t)\Phi(\xi) \tag{7}$$

$$\Theta_j(\xi) = \sum_{r=0}^{P} \theta_{j_r}\phi_r(\xi) = \theta_j^T\Phi(\xi) \tag{8}$$

$$L_{ij}(\Theta) = \sum_{r=0}^{P} L_{ij_r}\phi_r(\xi) = \mathbf{l}_{ij}^T\Phi(\xi) \tag{9}$$

$$M_{ij}(\Theta) = \sum_{r=0}^{P} M_{ij_r}\phi_r(\xi) = \mathbf{m}_{ij}^T\Phi(\xi) \tag{10}$$

where $\Phi(.) \in \mathbb{R}^P$ is a vector of polynomials basis functions orthogonal to the pdf $p(\xi)$, which can be constructed using the Gram–Schmidt orthogonalization process [26]. The coefficients L_{ij_r} and M_{ij_r} are obtained by making use of following *normal equations*:

$$L_{ij_r} = \frac{\langle L_{ij}(\Theta(\xi)), \phi_r(\xi)\rangle}{\langle \phi_r(\xi), \phi_r(\xi)\rangle}$$

$$M_{ij_r} = \frac{\langle M_{ij}(\Theta(\xi)), \phi_r(\xi)\rangle}{\langle \phi_r(\xi), \phi_r(\xi)\rangle}$$

where $\langle u(\xi), v(\xi)\rangle = \int_{\Omega} u(\xi)v(\xi)p(\xi)d\xi$ represents the inner product induced by pdf $p(\xi)$. For linear and polynomial functions, the integrals in the inner products can be easily evaluated analytically [11] to obtain the coefficients. For nonpolynomial nonlinearities, numerical quadrature methods are used to evaluate the multidimensional integrals.

The total number of terms in the gPC expansion is $P+1$ and is determined by the chosen highest order (l) of the polynomials $\{\phi_r\}$ and the dimension (m) of the vector of uncertain parameters Θ

$$P + 1 = \frac{(l+m)!}{l!m!} \tag{11}$$

Substitution of the approximate expressions for $\mathbf{x}$ and Θ in Eqs. (7) and (8), in Eq. (6) leads to

$$e_i(\xi) = \dot{\mathbf{z}}_i^T\Phi(\xi) - \sum_{j=1}^{N}\mathbf{l}_{ij}^T\Phi(\xi)\mathbf{z}_j^T(t)\Phi(\xi)$$

$$-\sum_{j=1}^{q}\mathbf{m}_{ij}^T\Phi(\xi)w_j(t), \text{ for } i = 1,\ldots,N \tag{12}$$

where $\mathbf{e}(\xi)$ represents the error due to the truncated gPC expansions of $\mathbf{z}$. The $N(P+1)$ time-varying unknown coefficients z_{ir}

can be obtained using the Galerkin projection method. Projecting the error onto the space of orthogonal basis functions $\{\phi_r\}$ and minimizing, it leads to $N(P+1)$ deterministic equations

$$\langle e_i(\xi), \phi_r(\xi)\rangle = 0 \quad \text{for} \quad i = 1, \ldots, N \text{ and } r = 0, \ldots, P$$

$$\dot{z}_{ir}\langle \phi_r^2\rangle - \left\langle \sum_{j=1}^{N} \mathbf{l}_{ij}^T \Phi \mathbf{z}_j^T(t)\Phi, \phi_r \right\rangle - \left\langle \sum_{j=1}^{q} \mathbf{m}_{ij}^T \Phi w_j(t), \phi_r \right\rangle = 0 \tag{13}$$

These integrals can be evaluated analytically for linear systems resulting in a set of deterministic ordinary differential equations (ODEs) with the coefficients of the gPC series expansions as the states

$$\dot{\mathbf{c}} = \mathbf{A}_p\mathbf{c} + \mathbf{B}_p\mathbf{w} \tag{14}$$

where $\mathbf{c}(t) = [\mathbf{z}_1^T(t), \mathbf{z}_2^T(t), \ldots, \mathbf{z}_N^T(t)]^T \in \mathbb{R}^{N(P+1)}$ is a vector of the gPC coefficients, $\mathbf{A}_p \in \mathbb{R}^{N(P+1)\times N(P+1)}$ is the deterministic system matrix, and $\mathbf{B}_p \in \mathbb{R}^{N(P+1)\times q}$ is the deterministic input matrix corresponding to the input vector $\mathbf{w}(t)$.

Let $T_r \in \mathbb{R}^{(P+1)\times(P+1)}$, for $r = 0, \ldots, P$, denote the inner product matrix of the orthogonal basis functions defined as follows:

$$\begin{aligned} T_{r_{ij}} &= \frac{1}{\langle \phi_r^2\rangle}\langle \phi_i(\xi)\phi_j(\xi), \phi_r(\xi)\rangle, \quad i,j = 0, \ldots, P \\ &= \frac{1}{\langle \phi_r^2\rangle}\int_\Omega \phi_i(\xi)\phi_j(\xi)\phi_r(\xi)d\xi \end{aligned} \tag{15}$$

Then, $\mathbf{A}_p$ can be written as an $N \times N$ matrix of block matrices, each block $A_{p_{ij}}$ being a $(P+1)\times(P+1)$ matrix the $(r)+)1)$th row of which is given by

$$A_{p_{ij}}(r, *) = \mathbf{l}_{ij}^T T_r, \quad i,j = 1, \ldots, N \tag{16}$$

The input matrix $\mathbf{B}_p$ can be written as $N \times q$ matrix of block matrices, each block $B_{p_{ij}}$ being a $(P+1)\times 1$ vector given by

$$B_{p_{ij}} = \mathbf{m}_{ij} \quad i = 1, \cdots, N, \quad j = 1, \ldots, q \tag{17}$$

Equation (14) can then be solved to obtain the time-history of the coefficients z_{ir}. This implies solving a system of $n(n+3)(P+1)/2$ simultaneous deterministic ordinary differential equations. The solution of the stochastic system in Eq. (6) can thus be obtained in terms of polynomial functionals of random variables ξ_i

$$z_i(t, \Theta) = \sum_{r=0}^{P} z_{ir}(t)\phi_r(\xi), i = 1, \ldots, N$$

This expression can be used to estimate the moments of the augmented state vector $\mathbf{z}$ consisting of various components for conditional mean $\mu(t, \Theta)$ and covariance matrix $\Sigma(t, \Theta)$

$$\mathbf{E}[z_i^m(t, \Theta)] = \mathbf{E}\left[\left(\sum_{r=0}^{P} z_{ir}(t)\phi_r(\xi)\right)^m\right], i = 1, \ldots, N \tag{18}$$

Now, the pdf of state vector $\mathbf{x}$ can be computed as follows:

$$\begin{aligned} p(\mathbf{x}) &= \int_\Omega p(t, \mathbf{x}|\Theta(\xi))p(\xi)d\xi \\ &= \int_\Omega \mathcal{N}(t, \mathbf{x}; \mathbf{z}(\xi))p(\xi)d\xi \\ &= \int_\Omega \mathcal{N}\left(t, \mathbf{x}; \sum_{r=0}^{P} z_{ir}(t)\phi_r(\xi)\right)p(\xi)d\xi \end{aligned} \tag{19}$$

The pdf given by the integral in Eq. (19) and its moments can be computed by making use of the quadrature integration scheme corresponding to the polynomial basis $\{\phi_r(\xi)\}$. For an orthogonal basis $\{\phi_r\}$ with $\phi_0 = 1$, the first two moments of the actual state vector $\mathbf{x}$ can be estimated analytically, as follows:

$$\begin{aligned} \mathbf{E}[x_i(t)] &= \int_\Omega \mathbf{E}[\mathcal{N}(t, \mathbf{x}; \sum_{r=0}^{P} z_{ir}(t)\phi_r(\xi))]p(\xi)d\xi \\ &= \int_\Omega \sum_{r=0}^{P} \mu_{ir}(t)\phi_r(\xi)p(\xi)d\xi = \mu_{i0}(t) \end{aligned} \tag{20}$$

$$\begin{aligned} \mathbf{E}[x_i^2(t)] &= \int_\Omega \mathbf{E}[x_i^2(t)|\xi]p(\xi)d\xi = \int_\Omega \left[\mu_i^2(t,\xi) + \Sigma_{ii}(t,\xi)\right]p(\xi)d\xi \\ &= \int_\Omega \left[\left(\sum_{r=0}^{P} \mu_{ir}(t)\phi_r(\xi)\right)^2 + \sum_{r=0}^{P} \Sigma_{ii_r}(t)\phi_r(\xi)\right]p(\xi)d\xi \\ &= \sum_{r=0}^{P} \mu_{ir}^2(t)\langle \phi_r^2\rangle + \Sigma_{ii_0}(t) \end{aligned} \tag{21}$$

A major advantage of this gPC based approach is that one can compute the sensitivity of mean and covariance of conditional pdf $p(t, \mathbf{x}|\Theta) = \mathcal{N}(t, \mathbf{x}; \mathbf{z}(\xi))$ with respect to unknown parameter vector $\Theta(\xi)$ while making use of Eqs. (14) and (19). Furthermore, one can use the sensitivity of conditional pdf to compute the sensitivity of the distribution of state vector $\mathbf{x}$ denoted by $p(\mathbf{x})$ with respect to unknown parameter vector $\Theta(\xi)$. While a detailed deterministic sensitivity analysis can relate the variations in input parameters to moments of state vector, *uncertainty analysis casts a much broader net in terms of assessing confidence of predictions based on all available information.*

3.2 Method 2: Conditioning First on Stochastic Forcing.

For a particular realization of the stochastic forcing terms, the uncertainty propagation in the linear model described by Eq. (1) can be described efficiently using the gPC approach. For a given $\eta = \eta(\omega)$, the model is similar to Eq. (6) and is given by

$$\dot{\mathbf{x}} = \mathbf{A}(\Theta)\mathbf{x} + \mathbf{B}(\Theta)\mathbf{u} + \mathbf{G}(\Theta)\eta(\omega) \tag{22}$$

Using the gPC approach, the solution of the system can be obtained in terms of approximate finite series expansion characterizing the conditional state pdf $p(\mathbf{x}|\eta)$, as follows:

$$x_i(t) = \sum_{r=0}^{P} x_{ir}(t,\omega)\phi_r(\xi) = \mathbf{x}_i^T(t,\omega)\Phi(\xi) \tag{23}$$

Realizations of the gPC coefficients x_{ir} are obtained for each realization of the Gaussian white noise process η. Each realization of the gPC coefficients characterizes the conditional distribution of $\mathbf{x}$ corresponding to a particular realization of the Gaussian white noise process. This conditional distribution is a function of the distribution $p(\xi)$ of the uncertain model parameters and is generally non-Gaussian. The moments of this conditional distribution can be estimated as in Eq. (18). Using the gPC method as described in Sec. 3.1.1 and following equations Eq. (7) through Eq. (14), the model in Eq. (22) with uncertain model parameters, is transformed into a system of equations with deterministic parameters as follows:

$$\dot{\mathbf{c}} = \mathbf{A}_p\mathbf{c} + \mathbf{B}_p\mathbf{u} + \mathbf{G}_p\eta(\omega) \tag{24}$$

where $\mathbf{c}(t) = [\mathbf{x}_1^T(t), \mathbf{x}_2^T(t), \ldots, \mathbf{x}_n^T(t)]^T \in \mathbb{R}^{n(P+1)}$ is a vector of the gPC coefficients, $\mathbf{A}_p \in \mathbb{R}^{n(P+1)\times n(P+1)}$ is the deterministic system matrix, $\mathbf{B}_p$ and $\mathbf{G}_p$ are the deterministic input matrices corresponding to the deterministic input $\mathbf{u}$ and stochastic input η, respectively. These matrices are obtained similarly as in Eqs. (16) and (17).

The model described in Eq. (24) is a deterministic linear model driven by a Gaussian white stochastic forcing η. Hence, the system states of this model are Gaussian, and the moments are exactly given by the moment propagation equations

$$\dot{\mu}_c = \mathbf{A}_p \mu_c + \mathbf{B}_p \mathbf{u} \tag{25}$$

$$\dot{\Sigma}_c = \mathbf{A}_p \Sigma_c + \Sigma_c \mathbf{A}_p^T + \mathbf{G}_p \mathbf{Q} \mathbf{G}_p^T \tag{26}$$

where $\mu_c = [\mu_1^T, \ldots, \mu_n^T]^T$ is the mean, and Σ_c is the covariance matrix of the vector of gPC coefficients $\mathbf{c}$. These moments characterize the Gaussian distribution of the gPC coefficients x_{ir}. Equations (25) and (26) require solving a system of $n(P+1)[n(P+1)+3]/2$ simultaneous deterministic ODEs. In terms of these moments, the pdf of state vector $\mathbf{x}$ can be computed as follows:

$$\begin{aligned} p(\mathbf{x}) &= \int_{-\infty}^{\infty} p(t, \mathbf{x} | \eta(\omega)) p(\omega) d\omega \\ &= \int_{-\infty}^{\infty} p\left(\sum_{r=0}^{P} x_{ir}(t)\phi_r(\xi)\right) \mathcal{N}(t, \mathbf{c}; \mu_c, \Sigma_c) d\mathbf{c} \end{aligned} \tag{27}$$

Quadrature techniques can be used to evaluate the integrals for estimating the moments of this distribution. For orthogonal basis $\{\phi_r\}$ with $\phi_0 = 1$, the first two moments of the state vector $\mathbf{x}$ can be estimated analytically as follows:

$$\begin{aligned} \mathbf{E}[x_i(t)] &= \int_{-\infty}^{\infty} \mathbf{E}\left[\sum_{r=0}^{P} x_{ir}(t,\omega)\phi_r(\xi)|\omega\right] \mathcal{N}(t, \mathbf{c}; \mu_c, \Sigma_c) d\mathbf{c} \\ &= \int_{-\infty}^{\infty} x_{i0}(t,\omega) \mathcal{N}(t, \mathbf{c}; \mu_c, \Sigma_c) d\mathbf{c} = \mu_{i0}(t) \end{aligned} \tag{28}$$

$$\begin{aligned} \mathbf{E}[x_i^2(t)] &= \int_{-\infty}^{\infty} \mathbf{E}\left[\left(\sum_{r=0}^{P} x_{ir}(t,\omega)\phi_r(\xi)\right)^2 |\omega\right] \mathcal{N}(t, \mathbf{c}; \mu_c, \Sigma_c) d\mathbf{c} \\ &= \int_{-\infty}^{\infty} \left(\sum_{r=0}^{P} x_{ir}^2(t,\omega)\langle\phi_r^2\rangle\right) \mathcal{N}(t, \mathbf{c}; \mu_c, \Sigma_c) d\mathbf{c} \\ &= \sum_{r=0}^{P} \left[\mu_{ir}^2(t) + \Sigma_{i_r i_r}^2(t)\right] \langle\phi_r^2\rangle \end{aligned} \tag{29}$$

3.3 Computational Comparison of the Methods. Method 1 requires solving a system of $n(n+3)(P+1)/2$ simultaneous deterministic ODEs. Method 2 requires solving a system of $n(P+1)[n(P+1)+3]/2$ simultaneous deterministic ODEs. A Monte Carlo approach for the same system requires solving the original system of equations given by Eq. (1), for many process noise and uncertain parameter realizations. For N Monte Carlo runs, the solution requires solving nN differential equations, which in general turns out to be much larger than the number of equations required for either of the proposed approaches. The computational time is expected to be the least for the first method and the largest for the Monte Carlo approach. Furthermore, it should be mentioned that the total number of gPC terms, $P+1$, increases factorially with the number of uncertain parameters as given by Eq. (11). This makes the implementation of gPC approach computationally expansive for systems with large number of unknown parameters. However, Non-intrusive spectral projection (NISP) [27–29] or Polynomial chaos quadrature (PCQ) [30] methods can be used to reduce some of the computational burden associated with the gPC approach.

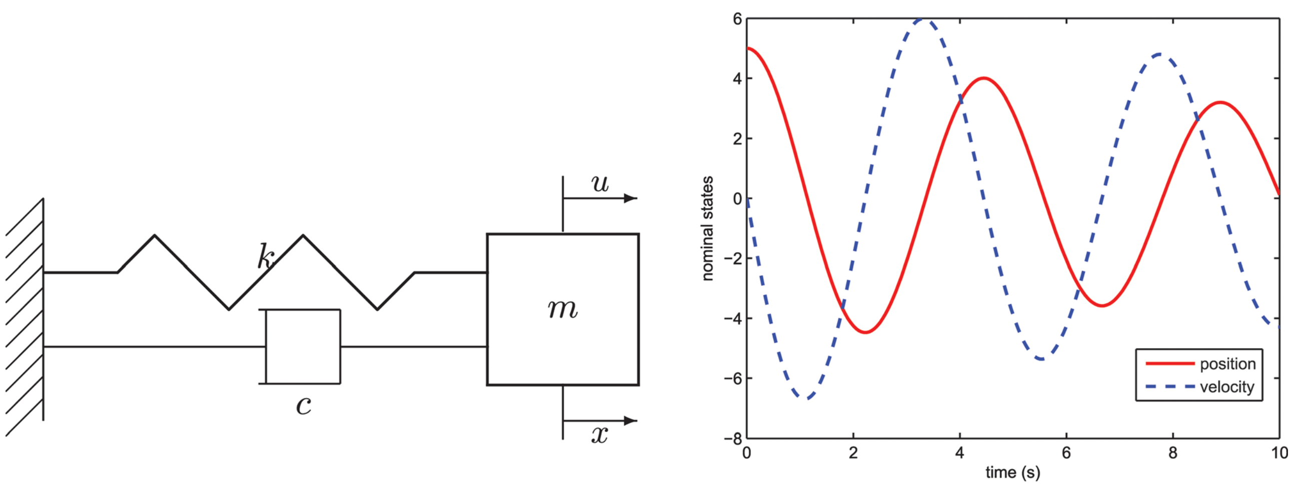
4 Results and Discussion

The proposed methods for uncertainty propagation are illustrated in this section using two numerical examples. The moments of the state probability distributions are evaluated using the two techniques proposed in this paper and are compared with the Monte Carlo solutions.

4.1 Spring-Mass System. A simple mass–spring–damper system, shown in Fig. 3(*a*), with an uncertain spring stiffness coefficient k, which is driven by a zero mean Gaussian stochastic forcing u is considered. The system is described by

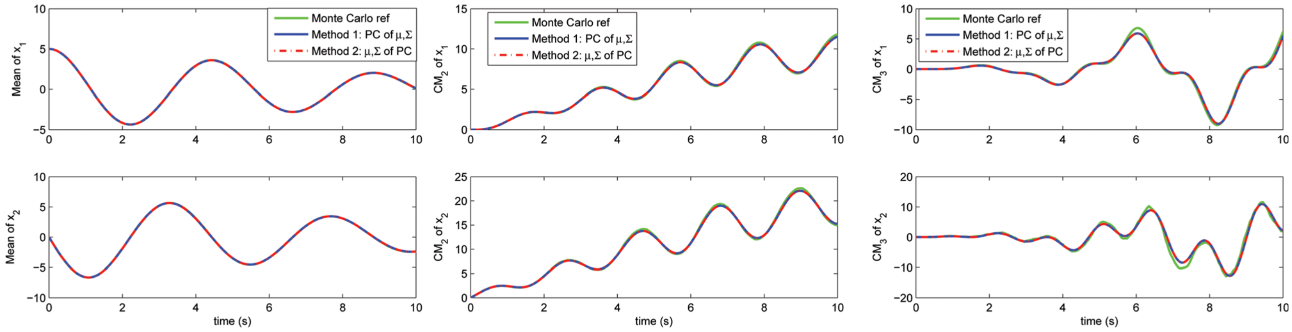
$$m\ddot{x} + c\dot{x} + kx = u \tag{30}$$

The mass, $m = 1$, is released at $x_0 = 5$, with velocity $\dot{x}_0 = 0$. The system has a known damping constant $c = 0.1$. The evolution of the states of the nominal system, for $k = 2$ and no stochastic forcing, is shown in Fig. 3(*b*). The spring stiffness k is assumed to be uniformly distributed between 1.5 and 2.5, while the zero mean Gaussian stochastic forcing has a standard deviation of 2. The true propagation of the uncertain states is well estimated by Monte Carlo solution of the uncertain system described by Eq. (30), with 10000 sample runs of the model, independently sampling the uncertain parameter k and the stochastic forcing u. The propagation of uncertainty in this linear dynamic model is estimated using the proposed approaches and compared with the reference Monte Carlo solution.



(*a*) Mass-spring-damper system (*b*) Evolution of nominal states

Fig. 3 Spring-mass system: (*a*) mass–spring–damper system and (*b*) evolution of nominal states



(a) Propagation of mean (b) Propagation of variance (c) Propagation of 3rd central moment

Fig. 4 Propagation of moments of the actual states: (*a*) propagation of mean, (*b*) propagation of variance, and (*c*) propagation of third central moment

In method 1 (Sec. 3.1), the given model is replaced by an augmented model whose states represent the uncertainty of the original states due to stochastic forcing. The conditional distribution of the states given the model parameters is Gaussian characterized by its mean and covariance. The conditional mean and covariance propagation equations represent the dynamics of this new augmented model, with the mean and covariance terms being the states. These states are then approximated by a gPC series expansion of order 7, accounting for the uncertainty in the model parameter k. Since k is a uniformly distributed random parameter, Legendre polynomials are used as the basis functions in the gPC approximation. The system is then transformed into a deterministic system of equations with the gPC coefficients of the conditional mean and covariance, as the new states. Solving this system involves the propagation of the gPC coefficients of the conditional mean and covariance of the actual states. In effect, the solution of this system represents the uncertainty of the states characterizing the uncertainty due to stochastic forcing.

In method 2 (Sec. 3.2), the states of the given model are first approximated by a gPC series expansion of order 7, representing the uncertainty in the model parameter k. The initial system is transformed into an expanded system with the gPC coefficients as the new states. The uncertainty of the gPC coefficients, due to the stochastic forcing, is then propagated using the moment propagation equations of the linear dynamic model. Solving this system involves the propagation of the mean and covariance of the gPC coefficients of the actual states. In effect, the output obtained represents the uncertainty of the states characterizing the uncertainty in model parameters.

The mean, variance (second central moment) and third central moment of the states, the position, and the velocity of the mass, are shown in Fig. 4. It can be seen that the estimated moments are consistent with that of the reference Monte Carlo solution. The histograms of the two states of the model at time $t = 10$ s are shown in Fig. 5, evaluated using the three approaches. Further, the run time for the Monte Carlo solution on a laptop gPC is 400 s, while the same for the solutions using the hybrid Bayesian-gPC methods are

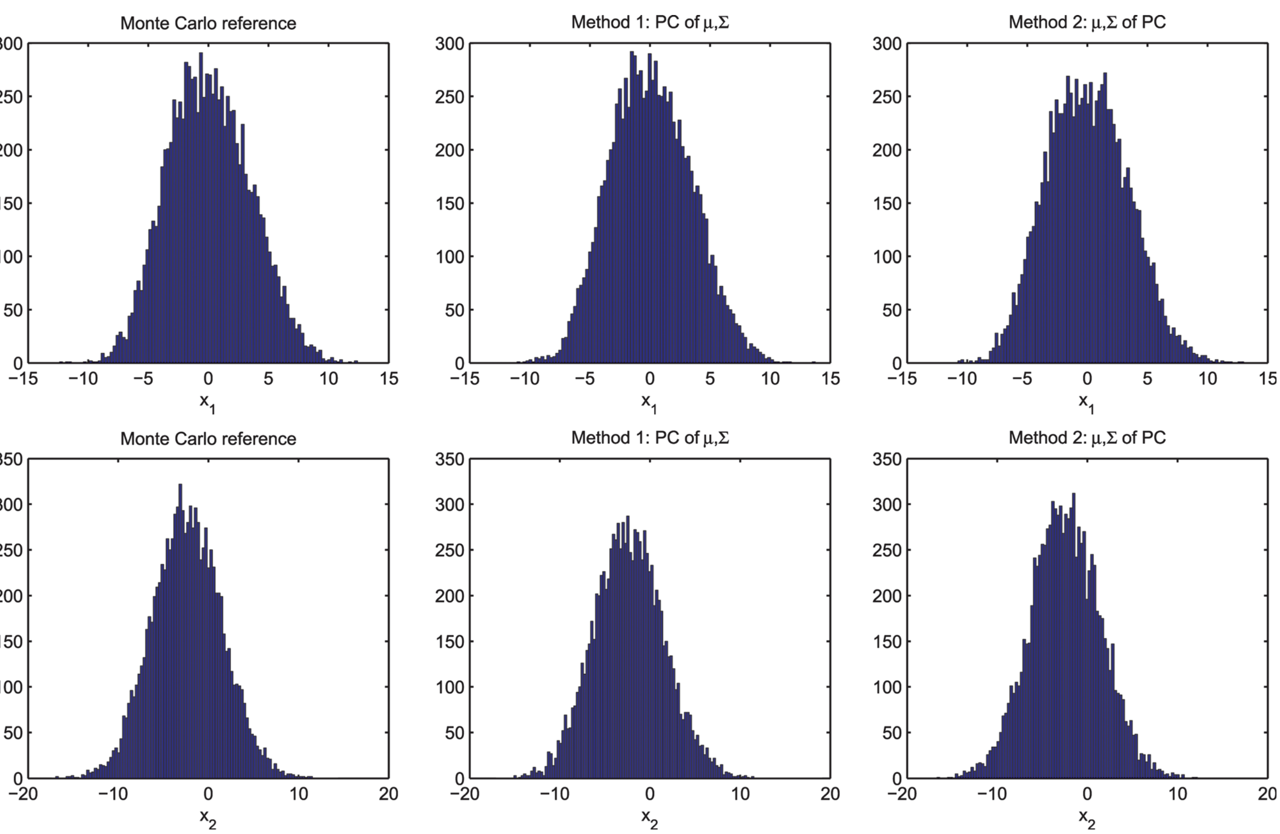
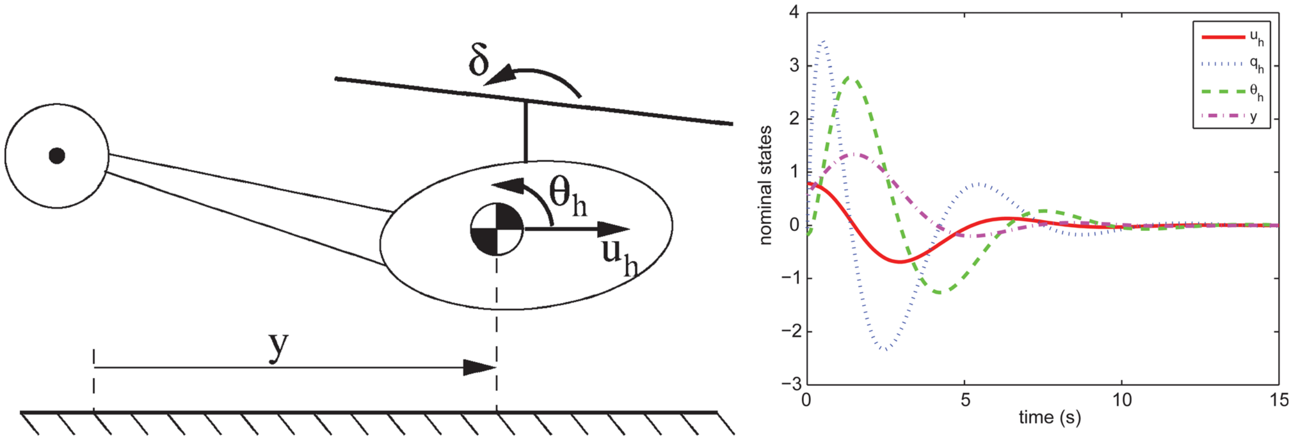


Fig. 5 Histograms of the states at $t = 10$ s



(*a*) Hovering helicopter [31]

(*b*) Evolution of nominal states

Fig. 6 Hovering helicopter system: (*a*) hovering helicopter [31] and (*b*) evolution of nominal states

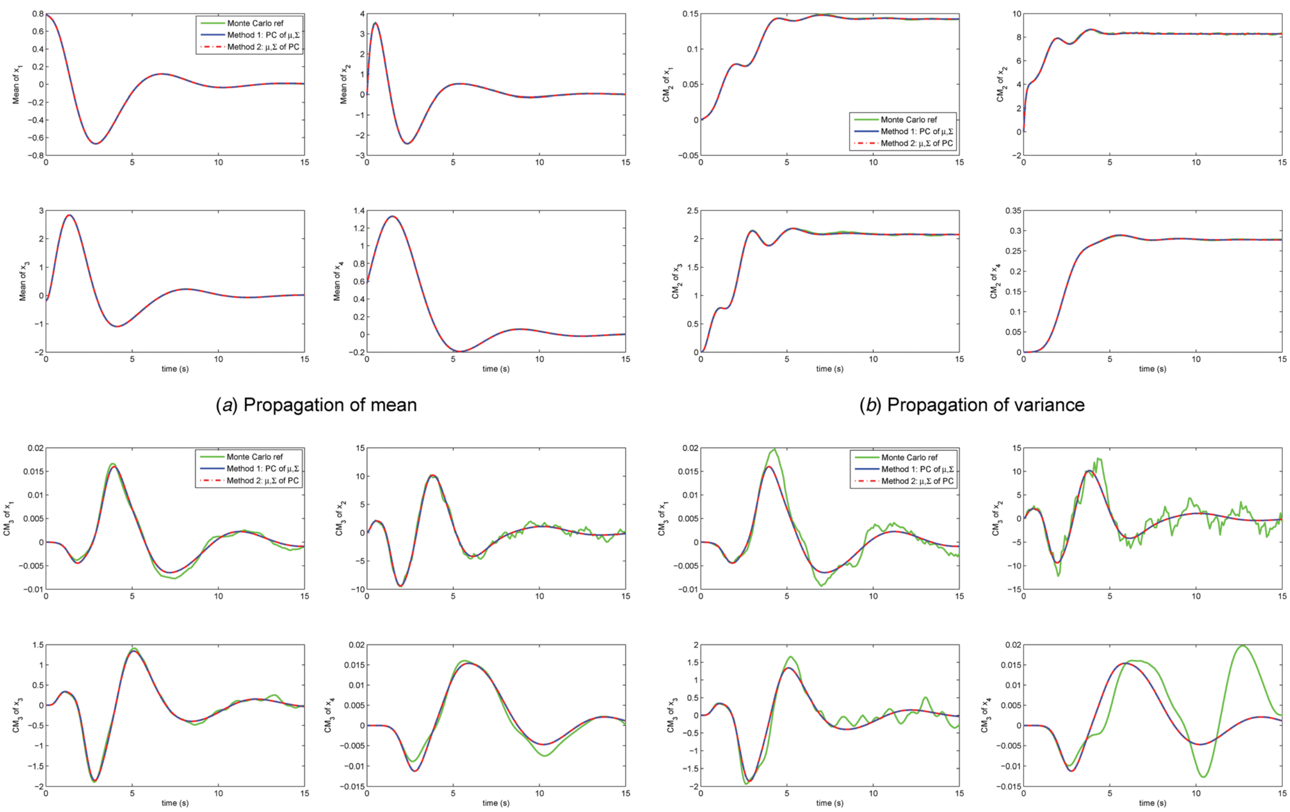
both less than 1 s. This illustrates the significant computational advantage of the proposed approaches over the Monte Carlo solution, while providing comparable results for this example.

4.2 Hovering Helicopter Model. The example of a hovering helicopter system [32] in the presence of random wind disturbance $u_w(t)$ and uncertainty in model parameters is considered to illustrate the significant computational advantage of the gPC based approaches. The decoupled approximation to the longitudinal motion of the OH-6A helicopter, shown in Fig. 6(*a*), is described by

$$\dot{\mathbf{x}} = A\mathbf{x} + B\delta + B_w u_w$$

$$\text{where,}\quad \mathbf{x} = \begin{bmatrix} u_h \\ q_h \\ \theta_h \\ y \end{bmatrix}, \quad A = \begin{bmatrix} p_1 & p_2 & -g & 0 \\ p_3 & p_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} p_5 \\ p_6 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B_w = \begin{bmatrix} -p_1 \\ -p_3 \\ 0 \\ 0 \end{bmatrix} \tag{31}$$



(*a*) Propagation of mean

(*b*) Propagation of variance

(*c*) Propagation of 3rd central moment

(*d*) 3rd central moment with 10000 Monte Carlo runs

Fig. 7 Propagation of moments of the states: (*a*) propagation of mean, (*b*) propagation of variance, (*c*) propagation of third central moment, and (*d*) third central moment with 10,000 Monte Carlo runs

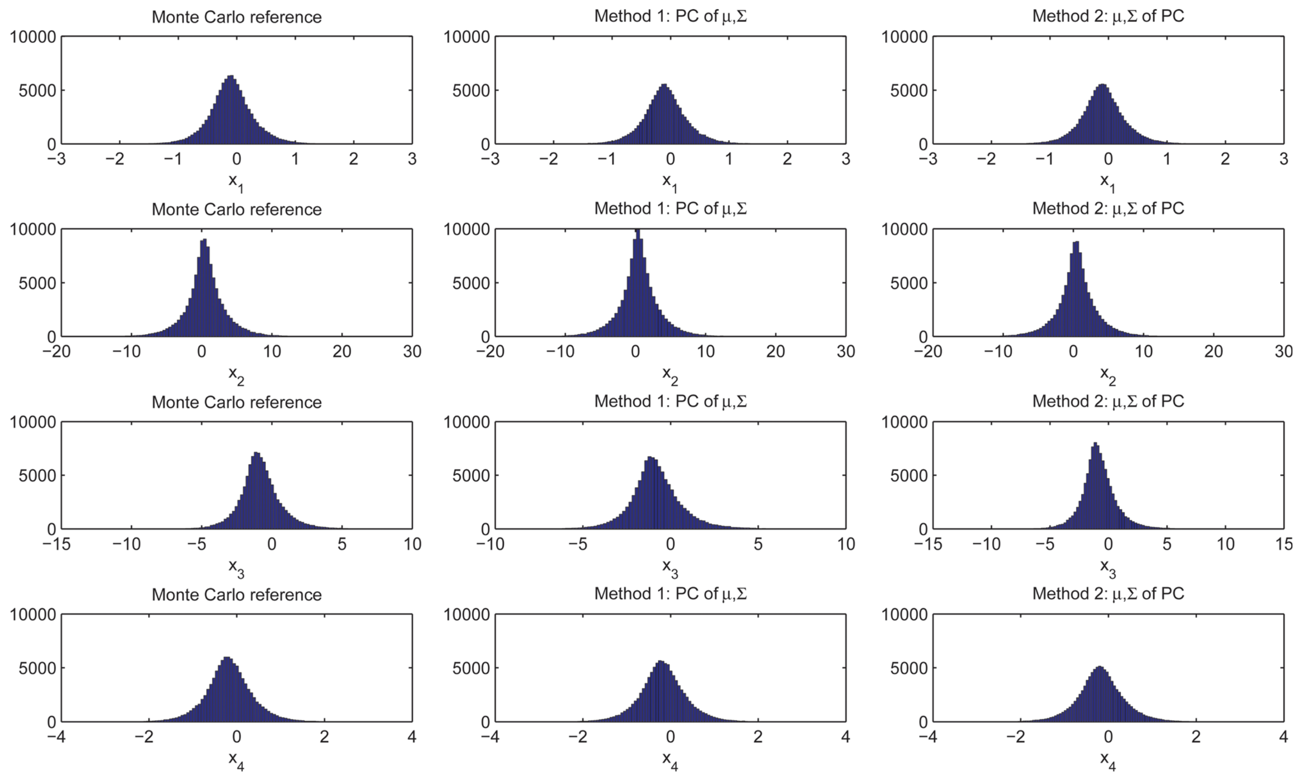


Fig. 8 Histograms of the states at $t = 10$ s

The state vector $\mathbf{x}$ describes the horizontal velocity u_h in foot per second, the pitch angle of the fuselage θ_h in centiradians, its derivative q_h in centi-radians per second, and perturbation y in foot from a ground point reference. g corresponds to the acceleration due to gravity given by 0.322, and u_w represents the wind disturbance described as a zero mean Gaussian white noise with a variance of $\sigma_w^2 = 18(ft/s)^2$. The six model parameters are referred to as the aerodynamic stability derivatives $(p_1, \ldots, p_4)$ and the aerodynamic control derivatives (p_5 and p_6). Let the unit initial condition vector of the model be given by

$$\mathbf{x}_0 = [\,0.7929 \quad -0.0466 \quad -0.1871 \quad 0.5780\,]^T \tag{32}$$

The evolution of the states of the model is shown in Fig. 6(b), for the nominal plant parameters

$$\mathbf{p} = [\,-0.0257 \quad 0.013 \quad 1.26 \quad -1.765 \quad 0.086 \quad -7.408\,] \tag{33}$$

and the control effort given by the longitudinal cyclic stick deflection δ in deci-inches

$$\delta = -Kx, \text{where } K = [\,1.9890 \quad -0.2560 \quad -0.7589 \quad 1.0000\,] \tag{34}$$

The four aerodynamic stability derivatives are assumed to be uncertain and uniformly distributed within the following bounds:

$$\begin{aligned} \mathbf{p}_{lb} &= [\,-0.0488,\ 0.0013,\ 0.126,\ -3.3535\,]^T, \\ \mathbf{p}_{ub} &= [\,-0.0026,\ 0.0247,\ 2.394,\ -0.1765\,]^T \end{aligned} \tag{35}$$

The true propagation of the uncertain states is estimated by Monte Carlo solution of the uncertain system described by Eq. (31), independently sampling the uncertain stability derivatives p_1, p_2, p_3 and p_4, and the wind disturbance u_w. In this example, for the gPC expansion, an order of 5 is chosen for both the approaches. The propagation of these uncertainties in this linear dynamic model is evaluated using the gPC based approaches and compared with the Monte Carlo solution.

The mean, variance and the third central moment of the four states of the helicopter model are shown in Fig. 7. In Figs. 7(a)–7(c), the Monte Carlo solution is obtained from 100000 runs. It can be seen that the estimated moments are consistent with that of the reference Monte Carlo solution. Further, the run time for the Monte Carlo solution on a laptop gPC is more than 1 h, compared to the method 1 run time of less than 5 s and 1 min for the method 2. In Fig. 7(d), the third central moments obtained from the gPC-based approaches are compared with that of the Monte Carlo solution obtained from 10000 runs. Comparing this with Fig. 7(c), it can be seen that the gPC solution is more accurate than the Monte Carlo solution with 10000 runs. The histograms of the four states of the model at time $t = 5$ s are shown in Fig. 8, computed using the three approaches. It can be seen that the histograms obtained from the different methods are comparable.

5 Conclusion

Two new efficient hybrid Bayesian approaches based on polynomial chaos are proposed in this work for the accurate determination of uncertainty propagation in linear dynamic models with parametric and initial condition uncertainties and driven by additive white Gaussian noise process. The uncertainty due to the AWGN stochastic forcing is propagated using mean and covariance propagation equations and that due to uncertain model parameters using polynomial chaos. While the moment propagation equations are exact only for white Gaussian stochastic forcing in linear dynamic models, the generalized polynomial chaos approach can be used for any probability distribution of model parameters. Both the proposed new methods are less computationally demanding than the standard Monte Carlo, which requires solving the dynamical model many times for many samples over the space of uncertain parameters.

These methods can be extended to nonlinear dynamic models, where the uncertainty due to white noise can be approximately propagated using model linearization, unscented transform or related methods. When measurements are available, this additional information about the distribution of the solution can be used to update propagation model-based predictions using prediction-correction filtering techniques. This suggests an extension of this approach to robust filtering problems with dynamical models having known parametric uncertainty distributions, a task currently under investigation.

Acknowledgment

This work is supported under Contract No. HM1582-08-1-0012 from ONR.

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