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Robust Minimum Power/Jerk Control of Maneuvering Structures

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The focus of this paper is on the development of weighted minimum power/jerk control profiles for the rest-to-rest maneuver of a flexible structure. To account for modeling uncertainties, equations, which represent the sensitivity of the system states to model parameters, are derived. The original state-space model of the flexible structure is augmented with the sensitivity state equations with the constraint that the sensitivity state variables are forced to be zero at the end of the maneuver. This requirement attenuates the residual vibration at the end of the maneuver caused by errors in system parameters. A systematic procedure for the design of the controller is developed by representing the linear-time-invariant system in its Jordan form. This decouples the modes of the system permitting us to address smaller-order dynamical systems. The proposed technique is illustrated via a benchmark floating oscillator problem.

I. Introduction

THE control of the benchmark two-mass/spring/damper system undergoing a rest-to-rest maneuver is to be considered in this paper. This problem, representative of many flexible structures, has one flexible mode and one rigid-body mode. A fairly comprehensive treatment of this family of problems has been presented by Junkins and Turner.[1] In previous research on this topic, time optimal control profiles have been derived by Singh et al.,[2] Ben-Asher et al.,[3] Farrenkopf,[4] and Hablani.[5] Desensitizing the control profiles to modeling errors has been addressed by Swigert,[6] Liu and Wie,[7] and Singh and Vadali.[8] Closed-form solutions have been obtained for the optimal control of the rest-to- rest maneuver using minimum power and minimum jerk cost functions by Bhat and Miu.[9,10] Minimum power solutions are obtained by minimizing $\int u^2\,\mathrm{d}\tau$, while minimum jerk solutions are obtained by minimizing $\int (\mathrm{d}u/\mathrm{d}t)^2\,\mathrm{d}\tau$. The u term in these cost functions is the control effort. Recently, it has been of interest to develop optimal solutions using a weighted cost function, such as the weighted fuel/time optimal control considered by Singh.[11] Here, the closed-form solution for the optimal control of the rest-to-rest maneuver using a weighted minimum power/jerk cost function is of interest. In the weighted cost function considered here, the user can select the relative importance of minimum power (or equivalently, minimum control effort) to minimum jerk (or equivalently, the minimum rate of change of control effort).

The solution for the control profile obtained for linear-time-invariant systems, like the system considered in this paper, often assumes known constant system parameters. With this assumption the simulated system response for a rest-to-rest maneuver will meet the required endpoint conditions with zero residual vibration. In actual physical systems it is impossible to know the exact values of the system parameters. Thus, any solution using the control profile obtained assuming constant system parameters will have zero residual vibration only when the actual system parameters exactly match the design parameters used to obtain the control profile. With this in mind, it is the goal of the researchers to obtain a solution that is robust to errors in system parameters (for example, damping ratio, natural frequency). To do this, sensitivity equations are derived and added to the state-space equations before transforming them into Jordan canonical form. It will be shown that with the addition of these equations, which force the sensitivity state variables to zero at the end of the maneuver, there is a reduction in residual vibration caused by errors in system parameters.

The paper begins with the problem formulation in Sec. II. Section III gives a numerical example with results presented. The topic of sensitivity equation formulation is considered in Sec. IV. Section V gives the same numerical example considered in Sec. III, with the addition of sensitivity equations. A comparison between the robust and nonrobust solutions is also drawn in this section. Finally, the paper concludes with a summary of the results obtained in Sec. VI.

II. Problem Formulation

The weighted minimum power/jerk cost function

$$\min \frac{1}{2}\int_0^T \left[\zeta^2 u^2 + \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2\right]\mathrm{d}\tau \tag{1}$$

is considered,[1] subject to the constraint

$$M\ddot{x} + \xi\dot{x} + Kx = Pu \tag{2}$$

where M is the mass matrix, ξ the damping matrix, and K is the stiffness matrix. P is the control influence vector, and u and x are the scalar control input and state vector, respectively. In Eq. (1) T is the specified final time, and ζ is the weighting parameter to be varied. The equations of motion for this system can be given in state-space form as

$$\dot{w} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}\xi \end{bmatrix}}_{A} w + \underbrace{\begin{bmatrix} 0 \\ M^{-1}P \end{bmatrix}}_{B} u \tag{3}$$

$$y = Cw + Du \tag{4}$$

Transforming this system of equations [Eqs. (3) and (4)] into Jordan canonical form gives

$$\dot{z} = J\underbrace{z}_{Vw} + \underbrace{b}_{VB}u \tag{5}$$

$$y = \underbrace{C^*}_{CV^{-1}}z + Du \tag{6}$$

where J is the Jordan canonical form of A and V is the transformation matrix. The solution of Eq. (5) is given as

$$e^{-Jt_2}z(t_2) - e^{-Jt_1}z(t_1) = \int_{t_1}^{t_2} e^{-J\tau}bu(\tau)\,\mathrm{d}\tau \tag{7}$$

To obtain the optimal control for this problem, calculus of variations will be used in order to perform the function optimization. Using this method, for the chosen cost function [Eq. (1)] to be minimized, the following performance criteria must be minimized:

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$$I = \frac{1}{2}\int_0^T \left[\zeta^2 u^2 + \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2\right]\mathrm{d}\tau$$

$$+ (\lambda^*)^T \left[e^{-JT} z(T) - \int_0^T e^{-J\tau} b u(\tau)\,\mathrm{d}\tau\right] \tag{8}$$

where λ^* in this equation represents the Lagrange multipliers. Equation (8) is derived by assuming the maneuver time to be T, the initial time and initial conditions to be zero, and by augmenting the cost function with Eq. (7). By taking the first variation of this equation and setting it equal to zero, the u that minimizes Eq. (1) can be obtained. The first variation is expressed as

$$\delta I = \int_0^T \left[\zeta^2 u - \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - (\lambda^*)^T e^{-J\tau} b\right]\delta u\,\mathrm{d}\tau + \left(\frac{\mathrm{d}u}{\mathrm{d}t}\delta u\right)_0^T \tag{9}$$

For this equation to be equal to zero for all δu, the quantities inside the brackets must be equal to zero. This requirement results in a differential equation in u, which is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - \zeta^2 u = -(\lambda^*)^T e^{-Jt} b \tag{10}$$

The control u in this equation can be determined by solving for the homogeneous and particular parts of this equation and then adding these solutions together to obtain the total solution for u. The boundary conditions for u are obtained from the necessary conditions for optimality. From Eq. (9), the conditions for optimality require that the second bracketed term must be equal to zero. Because δu is arbitrary, this implies that $\mathrm{d}u/\mathrm{d}t$ at the initial and final time must be equal to zero. There is no requirement that the control u be forced to zero at the initial and final time, as is done in the minimum jerk solution obtained in Ref. 10. In that paper the control is forced to zero at the initial and final times in order to make the control practical to input on a real physical system, although the cost function is not minimized. Thus, the minimum jerk solution obtained in Ref. 10 is suboptimal. Here, the optimal solution of the weighted minimum power/jerk solution will be considered, and it will be shown that as ζ goes to zero the solution converges to the optimal minimum jerk solution; conversely, the optimal minimum power solution is obtained as ζ goes to infinity.

A general closed-form solution for the weighted minimum power/jerk control can be found by solving Eq. (10) with the necessary condition for optimality that the derivative of the control at initial and final time is set equal to zero. The form of the solution will depend on the size of the system as well as the number of modes present. The general closed-form solution for a system with one rigid-body mode and p flexible modes, for $\zeta > 0$, is given as

$$u(t) = \lambda_1 + \lambda_2 t + \lambda_3 e^{\zeta t} + \lambda_4 e^{-\zeta t}$$

$$+ \sum_{i=1}^{p}\left[\lambda_{(3+2i)} e^{-a_i t}\sin(b_i t) + \lambda_{(4+2i)} e^{-a_i t}\cos(b_i t)\right] \tag{11}$$

where a_i is the real part of the ith complex conjugate pole and b_i is the imaginary part of the ith complex conjugate pole of the system. For $\zeta = 0$ the solution of Eq. (10) is

$$u(t) = \lambda_1 + \lambda_2 t + \lambda_3 t^2 + \lambda_4 t^3$$

$$+ \sum_{i=1}^{p}\left[\lambda_{(3+2i)} e^{-a_i t}\sin(b_i t) + \lambda_{(4+2i)} e^{-a_i t}\cos(b_i t)\right] \tag{12}$$

which corresponds to the minimum jerk solution. The parameters (λ_i) in Eq. (11) are found by simultaneously solving Eq. (7) and the boundary conditions from Eq. (9). The number of parameters (λ_i) in this solution is n, which in a general case depends on the size of the system. For the system considered here with one rigid-body mode, $n = 4 + 2p$, where the scalar p is the number of flexible modes of the system. The value obtained for n can be broken down as follows:

$$n = \underset{\text{(for 1 rigid-body mode)}}{2} + \underset{\text{(\# of B.C.s)}}{2} + \underset{\text{(for } p \text{ flexible modes)}}{2p} \tag{13}$$

$$= \text{dimension}(\lambda^*) + \underset{\text{(\# of B.C.s)}}{2} \tag{14}$$

The cost function used here is the same as used by Junkins and Turner.[1] Junkins and Turner convert the problem to a standard form for Pontryagin's Principle, whereas here the calculus of variations is used for the function optimization. Also, Junkins and Turner derive closed-form solutions for the case of rigid-body motion only, whereas a system with rigid-body and flexible modes is addressed in this paper. The procedure for obtaining the control using the method developed here will be demonstrated in the next section.

III. Numerical Example 1

The benchmark two-mass/spring/damper problem will now be considered. This system is a model of a flexible structure with one rigid-body mode and one flexible mode. Figure 1 shows the system to be considered, with the two masses m_0 and m_1, the spring constant k, and a viscous damper c. In the figure x_0 and x_1 are the displacement of the first and second mass, respectively. The input force is denoted as u and the output as y.

The differential equation that governs this system is given by

$$M\ddot{x} + \xi\dot{x} + Kx = Pu \tag{15}$$

where

$$M = \begin{bmatrix} m_0 & 0 \\ 0 & m_1 \end{bmatrix}, \qquad K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{16}$$

$$\xi = \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}, \qquad x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{17}$$

The linear ordinary coupled differential equation from Eq. (15) can be written in the state-space form as

$$\dot{w} = Aw + Bu \tag{18}$$

$$y = Cw + Du \tag{19}$$

where

$$w = \begin{bmatrix} x_0 \\ x_1 \\ \dot{x}_0 \\ \dot{x}_1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_0 & k/m_0 & -c/m_0 & c/m_0 \\ k/m_1 & -k/m_1 & c/m_1 & -c/m_1 \end{bmatrix} \tag{20}$$

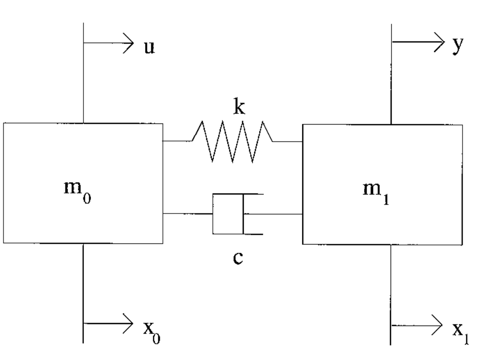

Fig. 1 Two-mass/spring/damper system.

$$B = \begin{bmatrix} 0 \\ 0 \\ 1/m_0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \qquad D = 0 \quad (21)$$

When the state-space equations are converted into Jordan canonical form, Eq. (18) can be rewritten as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}}_{Vw} + \underbrace{\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{bmatrix}}_{VB=b} u \quad (22)$$

where J is the Jordan canonical form of the A matrix, V is the transformation matrix, and p_1 and p_2 are the complex conjugate poles of the flexible mode.

The rest-to-rest maneuver of the undamped two-mass/spring/damper benchmark problem is considered here. For simplicity, the following parameters are used:

$$m_0 = m_1 = 1, \qquad k = \tfrac{1}{2}, \qquad c = 0, \qquad t_1 = 0 \text{ (initial time)}$$
$$t_2 = 4\pi \text{ (final time)}, \qquad \zeta = 1 \quad (23)$$

The input is on m_0, and the output is the position of m_1. The initial positions of m_0 and m_1 are both zero, and the final positions are chosen (arbitrarily) to be one. For this system the natural frequency of the elastic mode is equal to 1. Using the parameterized closed-form solution of u [Eq. (11)], the solution is found by rewriting Eq. (7) as

$$\begin{bmatrix} e^{-J(4\pi)} z(4\pi) \\ \cdots\cdots\cdots \\ 0 \\ 0 \end{bmatrix} = S\lambda \quad (24)$$

where S is given as

$$S = \text{Jacobian} \begin{bmatrix} \displaystyle\int_{t_1}^{t_2} e^{-J\tau} b u(\tau)\,\mathrm{d}\tau \\ \cdots\cdots\cdots \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(t_2) \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(t_1) \end{bmatrix} \text{ w.r.t. } \lambda \quad (25)$$

The S matrix for this undamped system under consideration is given analytically as

$$S = \begin{bmatrix} -8\pi^2 & -\dfrac{64}{3}\pi^3 & -\dfrac{4\pi\zeta e^{4\pi\zeta} - e^{4\pi\zeta} + 1}{\zeta^2} & \dfrac{4\pi\zeta e^{-4\pi\zeta} + e^{-4\pi\zeta} - 1}{\zeta^2} & 4\pi & 0 \\ 4\pi & 8\pi^2 & \dfrac{-1 + e^{4\pi\zeta}}{\zeta} & -\dfrac{-1 + e^{-4\pi\zeta}}{\zeta} & 0 & 0 \\ 0 & -4\sqrt{-1}\pi & \dfrac{-1 + e^{4\pi\zeta}}{\zeta + \sqrt{-1}} & \dfrac{-e^{-4\pi\zeta} + 1}{\zeta - \sqrt{-1}} & 2\sqrt{-1}\pi & 2\pi \\ 0 & 4\sqrt{-1}\pi & \dfrac{-1 + e^{4\pi\zeta}}{\zeta - \sqrt{-1}} & \dfrac{-e^{-4\pi\zeta} + 1}{\zeta + \sqrt{-1}} & -2\sqrt{-1}\pi & 2\pi \\ 0 & 1 & \zeta e^{4\pi\zeta} & -\zeta e^{-4\pi\zeta} & 1 & 0 \\ 0 & 1 & \zeta & -\zeta & 1 & 0 \end{bmatrix} \quad (26)$$

From Eq. (24) the unknown λ vector is determined using

$$\lambda = S^{-1} \begin{bmatrix} e^{-J(4\pi)} z(4\pi) \\ \cdots\cdots\cdots \\ 0 \\ 0 \end{bmatrix} \quad (27)$$

Using the numerical values for this example, λ is given as

$$\lambda = \begin{bmatrix} 0.106437 \\ -0.016940 \\ 0.15E{-6} \\ -0.043842 \\ -0.026902 \\ 0 \end{bmatrix} \quad (28)$$

These λ values are substituted into Eq. (11), and the solution is obtained. The control for the rest-to-rest maneuver given in this example is then

$$u(t) = 0.106437 - 0.01694t + (0.15E{-6})e^{t}$$
$$- (0.043842)e^{-t} - 0.026902 \sin(t) \quad (29)$$

It can be noted from Eq. (26) that there exist large variations in the magnitude of the elements of the S matrix, which might lead to numerical instabilities because the S matrix requires inversion. This is as a result of the $e^{\zeta t}$ term in the control profile, which can become inordinately large for large maneuver times. This can be remedied by normalization of the maneuver time to one.

Figure 2 is a plot of the control profile [Eq. (29)] and the position of both masses (m_0 and m_1) using the values chosen in Eq. (23). The rest-to-rest maneuver is completed without any residual vibration.

The next point of interest is the effect varying the weighting parameter ζ has on the values of λ. First, the general form of the control [Eq. (11)] for this example is given as

$$u(t) = \lambda_1 + \lambda_2 t + \lambda_3 e^{\zeta t} + \lambda_4 e^{-\zeta t} + \lambda_5 e^{-at}\sin(bt) + \lambda_6 e^{-at}\cos(bt) \quad (30)$$

where

$$a = 0, \qquad b = 1 \quad (31)$$

With these values of λ clearly defined, the effect of varying ζ on the parameters can be determined and is illustrated in Figs. 3–5. The parameter λ_6 is not shown because it is zero or negligible for all ζ values for this undamped system under consideration. The smallest value of ζ plotted for all cases is $\zeta = 0.01$, which is chosen to be a small number greater than zero because the weighted solution is not valid for $\zeta = 0$. Recall that in the case of $\zeta = 0$, the minimum jerk solution is obtained [Eq. (12)].

It will be shown here that as ζ goes to zero the weighted minimum power/jerk control profile converges to the minimum jerk control profile. This will be shown for the example system considered here. The convergence of the weighted minimum power/jerk control to the minimum power control as ζ tends toward infinity will also be

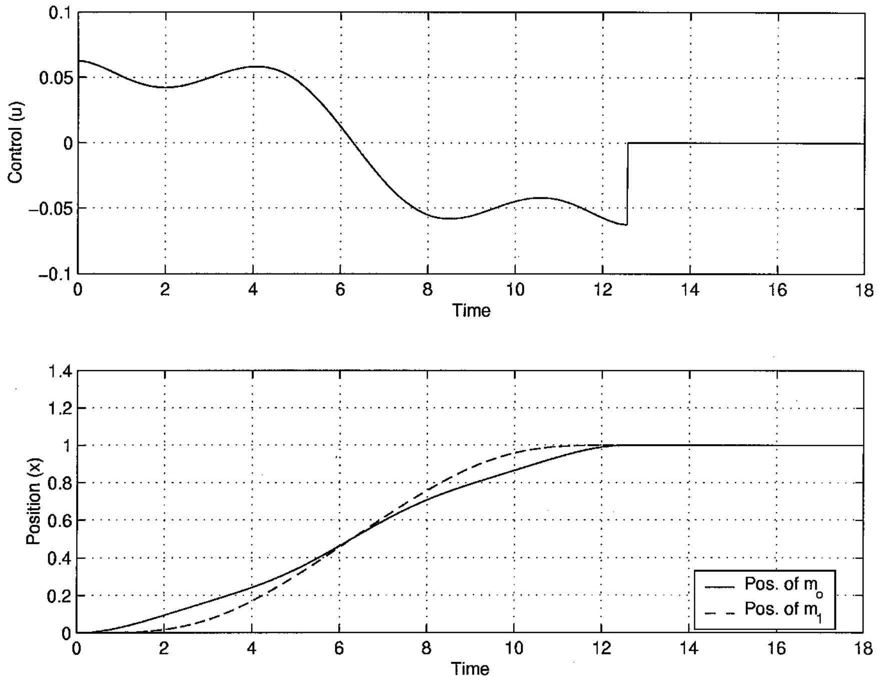

Fig. 2 Undamped system minimum power/jerk rest-to-rest maneuver.

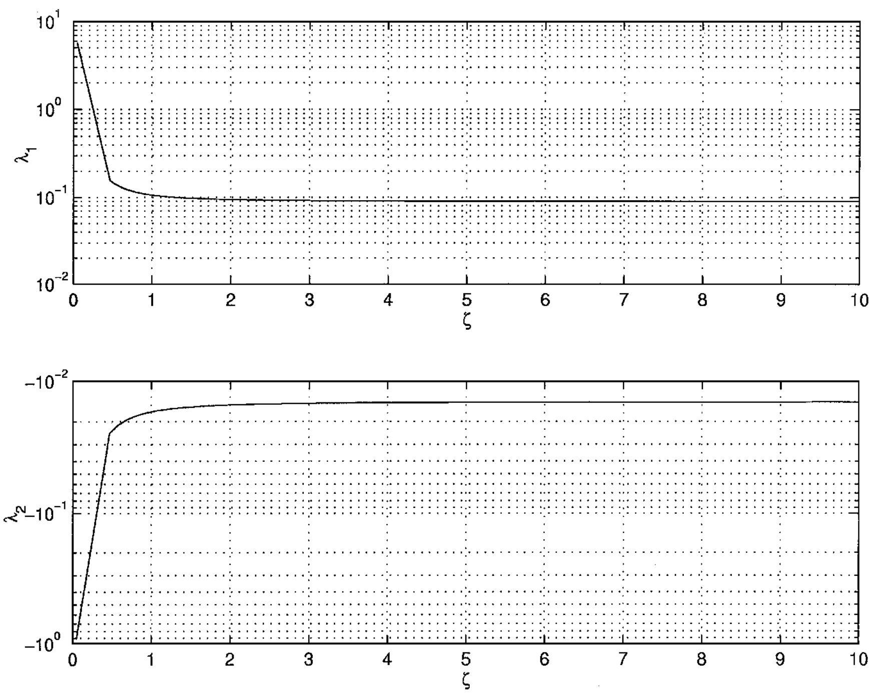

Fig. 3 λ_1 and λ_2 vs weighting parameter ζ for undamped system minimum power/jerk control.

demonstrated for this two-mass/spring/damper example. Before the convergence of the solution in the limits of ζ can be shown, the minimum power and minimum jerk solutions must be stated. The minimum power control profile for this system, which is determined using the closed-form minimum power solution given by Bhat and Miu,[10] is

$$u(t) = 0.08961 - 0.01426t - 0.02852\ \sin(t) \tag{32}$$

while the minimum jerk solution is

$$u(t) = 0.05017 + 0.02026t - 0.00674t^2 + 0.00036t^3 - 0.02026\ \sin(t) \tag{33}$$

The minimum jerk solution is obtained using the same procedure as in Ref. 10, except that the derivative of the control is forced to zero at initial and final time. Bhat and Miu[10] require the control to be zero at initial and final times, resulting in a suboptimal solution. These solutions are shown here for demonstration of the convergence of the weighted minimum power/jerk to the minimum power and minimum jerk solutions in the limits of the weighting parameter ζ. Figures 6 and 7 show the convergence of the weighted minimum power/jerk solution in the limits of ζ. A ζ value of 10 is chosen to demonstrate the convergence of the weighted minimum power/jerk solution to the minimum power solution as ζ goes to infinity. Figure 6 plots the solution obtained for this case along with the minimum power solution given by Eq. (32). A ζ value of 0.005 is chosen to demonstrate the convergence of the weighted minimum

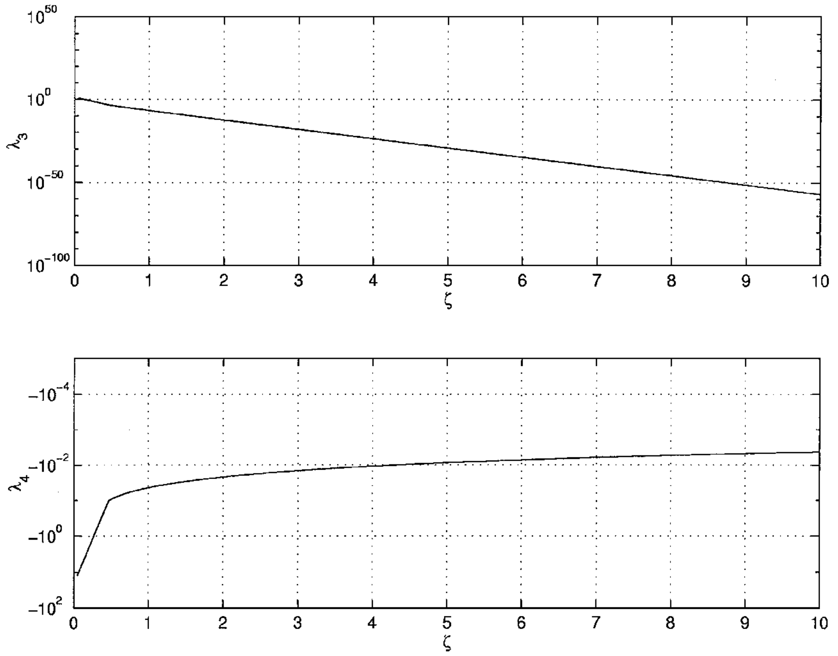

Fig. 4 λ_3 and λ_4 vs weighting parameter ζ for undamped system minimum power/jerk control.

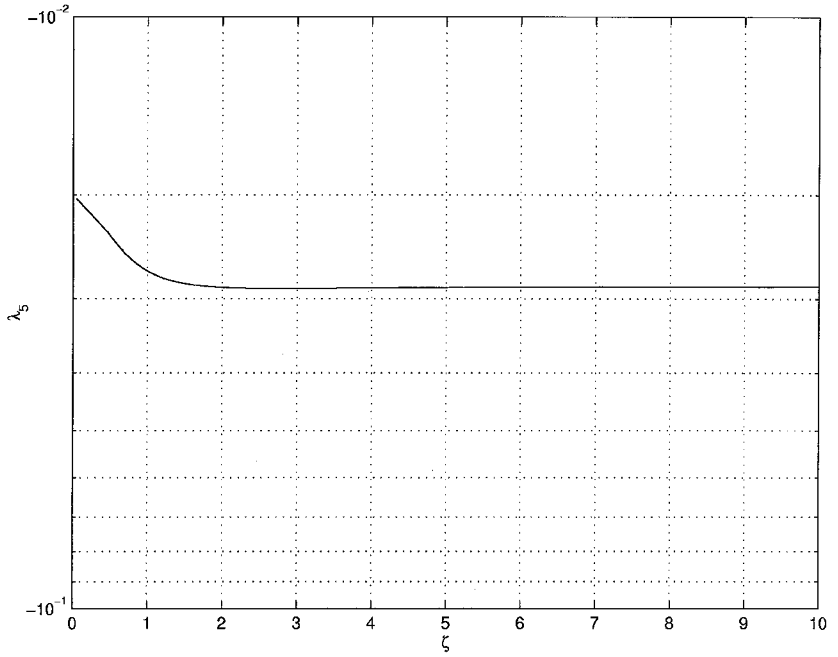

Fig. 5 λ_5 vs weighting parameter ζ for undamped system minimum power/jerk control.

power/jerk solution to the minimum jerk solution as ζ goes to zero. Figure 7 plots the solution obtained for this case along with the minimum jerk solution of Eq. (33). In both Figs. 6 and 7 there appears to be only one curve because the two curves in each plot lie on top of each other.

Following the same procedure used to determine the control in Fig. 2, the control for the damped benchmark two-mass/spring/damper problem can be obtained. Using the values given in Eq. (23) for this system, this time with $c = 0.25$ instead of $c = 0$, the control for the rest-to-rest maneuver is found to be

$$u(t) = 0.108260 - 0.019430t + (0.40E{-}6)e^{t} - (0.023246)e^{-t} - 0.003277e^{(t/4)}\sin\left[\left(\sqrt{15}/4\right)t\right] - 0.002571e^{(t/4)}\cos\left[\left(\sqrt{15}/4\right)t\right] \tag{34}$$

where for this case

$$a = -\tfrac{1}{4} \tag{35}$$

$$b = \sqrt{15}/4 \tag{36}$$

Figure 8 is a plot of the control profile [Eq. (34)] and the position of both masses (m_0 and m_1) for the system using the values given for this damped system. As with the undamped case, the rest-to-rest maneuver is completed without any residual vibration.

This section has demonstrated the validity of the weighted minimum power/jerk control determination as applied to a rest-to-rest maneuver of a flexible structure, with and without the inclusion of damping. The next section extends this idea to develop a robust solution when there are errors present in system parameters.

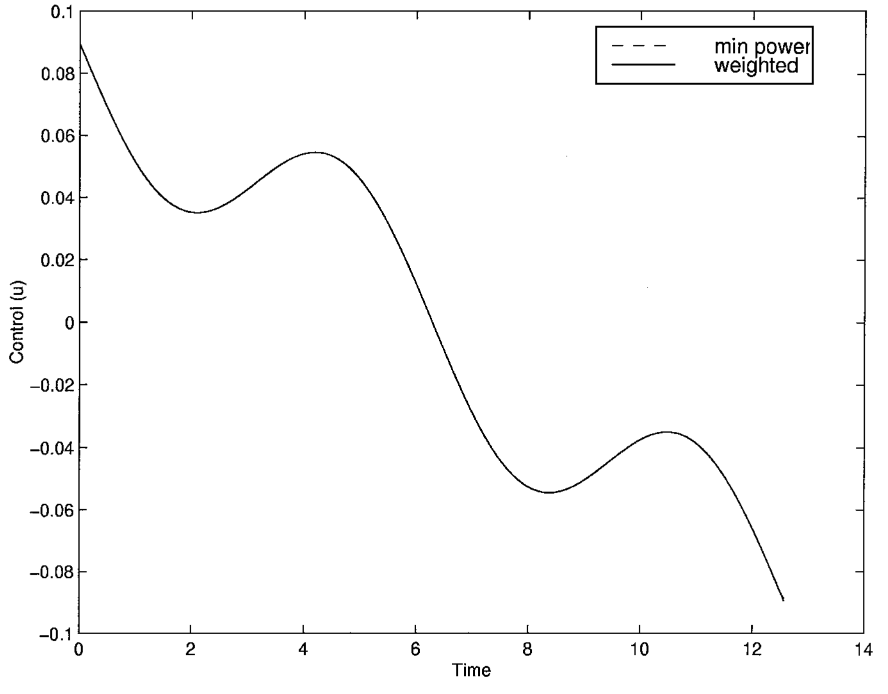

Fig. 6 Minimum power control vs weighted minimum power/jerk control when ζ = 10.

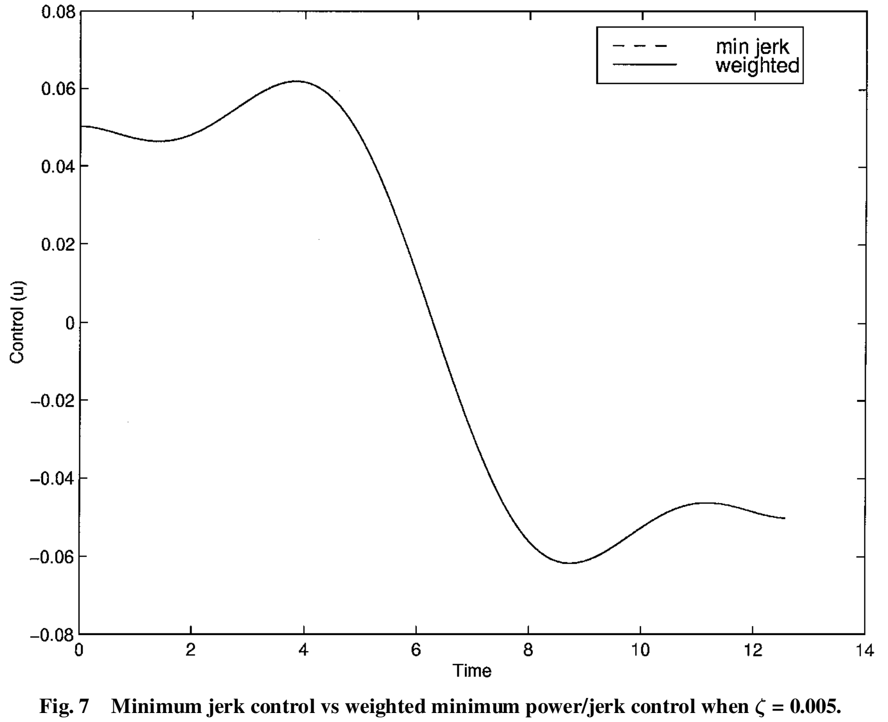

Fig. 7 Minimum jerk control vs weighted minimum power/jerk control when ζ = 0.005.

IV. Robust Solution — Sensitivity Equations

The goal of this paper is to formulate a control that minimizes a weighted minimum power/jerk cost function while being robust to errors in system parameters. The motivation comes from the fact that in real physical systems the values which are assumed to be constant (e.g., stiffness or natural frequency) are not known exactly or have been estimated. When this is the case, for the rest-to-rest maneuver considered here the control profile determined using known constant system parameters will bring the system to rest only when the actual system parameters are exactly the same as those assumed to obtain the control profile. When the actual system parameters are not the same, residual vibration will exist at the end of the rest-to-rest maneuver. The goal is to minimize this residual vibration. To do this, sensitivity equations are derived, which represent the sensitivity of the system to model parameters. This procedure can be applied to desensitize the system with respect to the stiffness k, the damping c, or the natural frequency (taking into consideration both errors in mass and stiffness). Here, the sensitivity with respect to the stiffness is of interest (thereby the natural frequency). It will be shown that the control profile obtained with the addition of these sensitivity equations reduces residual vibration when errors in the value of k (stiffness) are present. The equations of motion for the benchmark two-mass/spring/damper problem considered in Sec. III are

$$\ddot{x}_0 = (-k/m_0)x_0 + (k/m_0)x_1 + (-c/m_0)\dot{x}_0 + (c/m_0)\dot{x}_1 + u/m_0 \tag{37}$$

$$\ddot{x}_1 = (k/m_1)x_0 + (-k/m_1)x_1 + (c/m_1)\dot{x}_0 + (-c/m_1)\dot{x}_1 \tag{38}$$

By taking the derivative of these two equations with respect to k, the following equations are obtained:

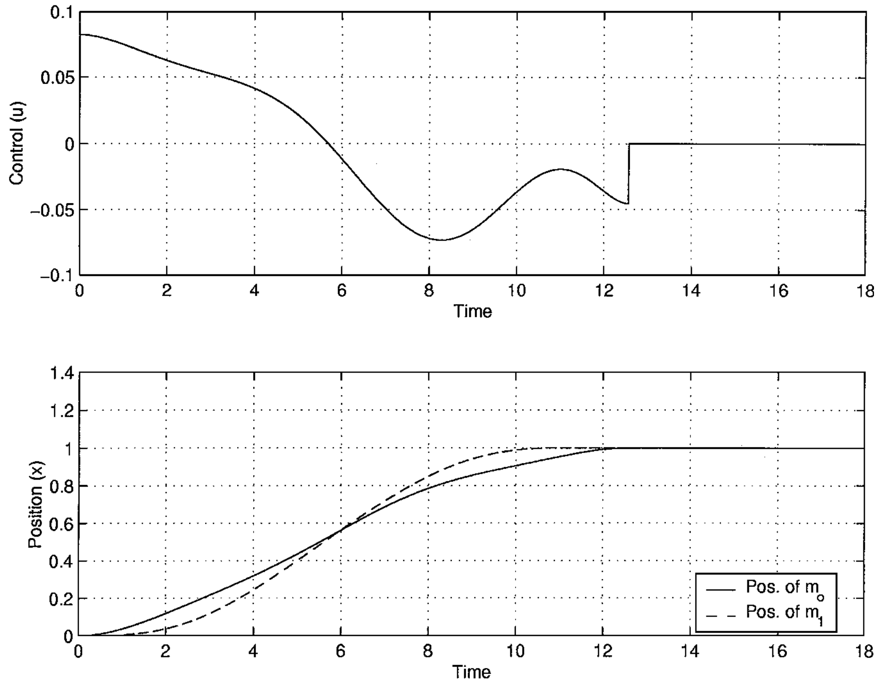

Fig. 8 Damped system minimum power/jerk rest-to-rest maneuver.

$$\frac{d\ddot{x}_0}{dk}+\frac{k}{m_0}\left(\frac{dx_0}{dk}-\frac{dx_1}{dk}\right)+\frac{c}{m_0}\left(\frac{d\dot{x}_0}{dk}\right)$$

$$+\frac{-c}{m_0}\left(\frac{d\dot{x}_1}{dk}\right)+\frac{x_0}{m_0}-\frac{x_1}{m_0}=0 \tag{39}$$

$$\frac{d\ddot{x}_1}{dk}-\frac{k}{m_1}\left(\frac{dx_0}{dk}-\frac{dx_1}{dk}\right)+\frac{-c}{m_1}\left(\frac{d\dot{x}_0}{dk}\right)$$

$$+\frac{c}{m_1}\left(\frac{d\dot{x}_1}{dk}\right)-\frac{x_0}{m_1}+\frac{x_1}{m_1}=0 \tag{40}$$

To simplify formulation while still demonstrating the benefit of this method, the values of the two masses are assumed to be equal as in the preceding example ($m_0 = m_1$). This does not have to be the case, but it makes for simple understanding of the procedure. If this assumption is not made, instead of having the single sensitivity equation given in Eq. (44), Eqs. (39) and (40) would represent the sensitivity equations with which the original state-space model would be augmented. Using the equal mass assumption, the following equation is obtained using Eqs. (39) and (40):

$$\frac{dx_0}{dk}=-\frac{dx_1}{dk} \tag{41}$$

Substituting into Eq.(39) gives

$$\frac{d\ddot{x}_0}{dk}+\frac{c}{m}\left(2\frac{d\dot{x}_0}{dk}\right)+\frac{k}{m}\left(2\frac{dx_0}{dk}\right)+\frac{x_0}{m}-\frac{x_1}{m}=0 \tag{42}$$

Defining a new state variable by

$$\frac{dx_0}{dk}=x_2 \tag{43}$$

gives the sensitivity equation, from Eq. (42), to be

$$\ddot{x}_2+2(c/m)\dot{x}_2+2(k/m)x_2+x_0/m-x_1/m=0 \tag{44}$$

Now, this equation will be added to the dynamical equations of the system and written in state-space form. The new system equations are (assuming $m_0 = m_1$)

$$\dot{w}=Aw+Bu \tag{45}$$

$$y=Cw+Du \tag{46}$$

where

$$A=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\-k/m&k/m&0&-c/m&c/m&0\\k/m&-k/m&0&c/m&-c/m&0\\-1/m&1/m&-2k/m&0&0&-2c/m\end{bmatrix} \tag{47}$$

$$B=\begin{bmatrix}0\\0\\0\\1/m\\0\\0\end{bmatrix},\qquad C=\begin{bmatrix}0&1&0&0&0&0\end{bmatrix},\qquad D=0 \tag{48}$$

$$w=\begin{bmatrix}x_0\\x_1\\x_2\\\dot{x}_0\\\dot{x}_1\\\dot{x}_2\end{bmatrix} \tag{49}$$

Once the sensitivity equations are added and the system equations are placed in state-space form, the same procedure used to determine the control (from Sec. II) can be used. It can be shown for this case that the same control profile will be obtained if the sensitivity is taken with respect to the damping c. This implies that desensitizing the system with respect to stiffness simultaneously desensitizes the system with respect to damping. By forcing the sensitivity states to zero at final time, the residual vibration is reduced. The example considered in Sec. III will be considered in the next section, this time with the addition of the derived sensitivity equations. A general closed-form solution for the robust weighted minimum power/jerk control can be found by solving Eq. (10) with the necessary condition

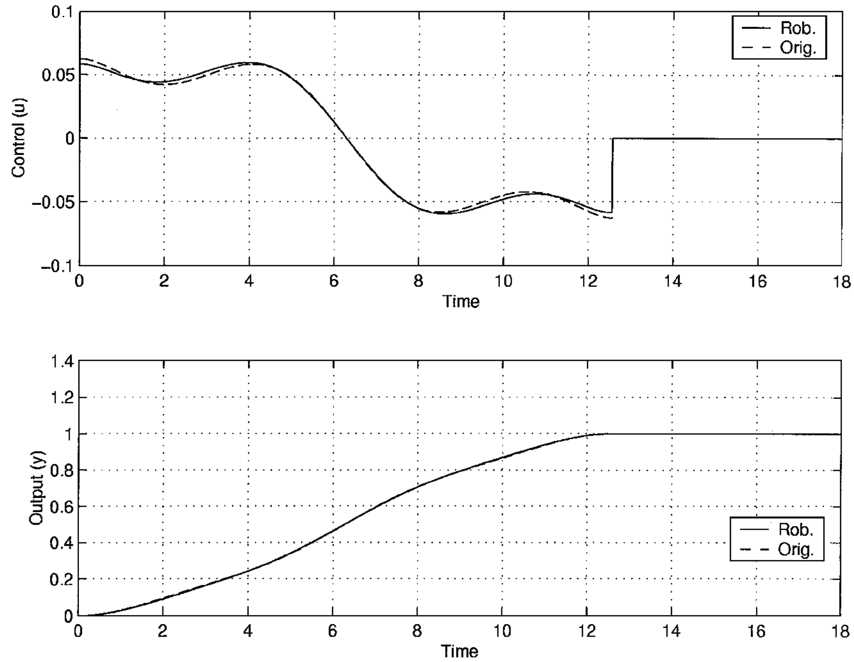

Fig. 9 Undamped system nonrobust and robust control without error in *k*.

for optimality that the derivative of the control at initial and final time is set equal to zero. Also, the J matrix given in Eq. (10) must be augmented with the sensitivity equations to analytically obtain the robust minimum power/jerk closed-form solution given in Eq. (50). The form of the solution will depend on the size of the system as well as the number of modes present. The general closed-form solution for the benchmark problem having one rigid-body mode and p flexible modes with the addition of the derived sensitivity equations is given as

$$u(t) = \lambda_1 + \lambda_2 t + \lambda_3 e^{\zeta t} + \lambda_4 e^{-\zeta t} + \sum_{i=1}^{p} \Big[\lambda_{(1+4i)} e^{-a_i t} \sin(b_i t) + \lambda_{(2+4i)} e^{-a_i t} \cos(b_i t) + \lambda_{(3+4i)} t e^{-a_i t} \sin(b_i t) + \lambda_{(4+4i)} t e^{-a_i t} \cos(b_i t)\Big] \tag{50}$$

where a_i is the real part of the ith complex conjugate pole and b_i is the imaginary part of the ith complex conjugate pole of the system. This general closed-form solution is only valid when the sensitivity is taken with respect to the stiffness (or damping) for the system considered. The next section uses this closed-form solution to obtain the weighted minimum power/jerk control for the example discussed in Sec. III with the inclusion of the sensitivity equations derived here. Both the undamped and damped systems will be addressed.

V. Numerical Example 2

Using the general closed-form solution given in Eq. (50), the robust weighted minimum power/jerk control will be determined for the benchmark problem already considered with system values given by Eq. (23) and the addition of the sensitivity equation derived in the preceding section. First the undamped system will be considered, followed by the damped system. Following the same procedure used in Sec. III, the following λ values are given for the undamped system:

$$\lambda = \begin{bmatrix} 0.106792 \\ -0.016996 \\ 0.14E{-}6 \\ -0.042876 \\ -0.026739 \\ -0.005400 \\ 0 \\ 0.000859 \end{bmatrix} \tag{51}$$

and the control is given as

$$u(t) = 0.106792 - 0.016996t + (0.14E{-}6)e^{t} - (0.042876)e^{-t} - 0.026739 \sin(t) - 0.005400 \cos(t) + 0.000859t \cos(t) \tag{52}$$

Figure 9 is a plot of the control (input) and position of m_1 (output) for both the nonrobust solution [Eq. (29)] and robust solution [Eq. (52)] when there are no errors in the system parameter k. Both solutions reach the final position without any residual vibration. Figure 9 does not show the position of m_0 for clarity, but it can be inferred from the displacement of m_1 that the residual energy of the system after the completion of the maneuver is zero because the two masses are statically coupled. In the following plots the position of m_0 will not be shown for the aforementioned reason.

Figure 10 is a plot of the same control input for the nonrobust and robust solutions shown in Fig. 9, this time when there is a 20% high error in $k[(k = 1.2 * (\frac{1}{2})]$. The figure shows a reduction in the residual vibration as a result of the error in the system parameter k with the robust solution.

Figure 11 is a plot of the residual energy in the system at the specified final time vs the actual system k value using a design k value of $\frac{1}{2}$ for both the nonrobust and robust solutions. The residual energy in the system at the final time is a scalar quantity defined as

$$\text{residual energy} = \sqrt{\tfrac{1}{2}\left(\underline{x}_r K \underline{x}_r^T + \underline{\dot{x}} M \underline{\dot{x}}^T\right)} \tag{53}$$

where K is the stiffness matrix, M is the mass matrix, $\underline{\dot{x}}$ is the vector of velocities, and $\underline{x}_r$ is a vector of residual positions defined as

$$\underline{x}_r = \underline{x}(t_2) - \underline{x}_{\text{desired}}(t_2) \tag{54}$$

From this figure the robust solution for this system reduces residual vibration when the actual system k values are less than approximately 0.405 and/or greater than approximately 0.48, with the exception that both the nonrobust and robust solutions will have zero residual vibration at the design value of $k = \frac{1}{2}$. In the region where $0.405 \le k \le 0.48$, the nonrobust solution has a smaller residual vibration as compared to the robust solution.

Next, the robust solution for the damped system considered in Sec. III will be considered. The values for this system are given by Eq. (23) with the value of $c = 0.25$ instead of $c = 0$ for this damped

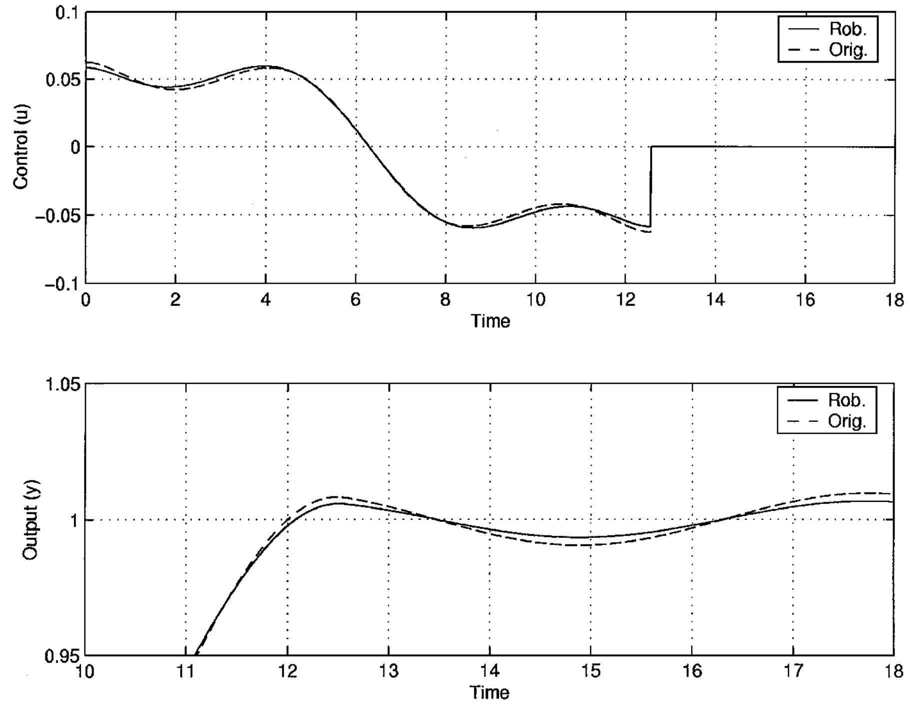

Fig. 10 Undamped system nonrobust and robust control with 20% error in *k*.

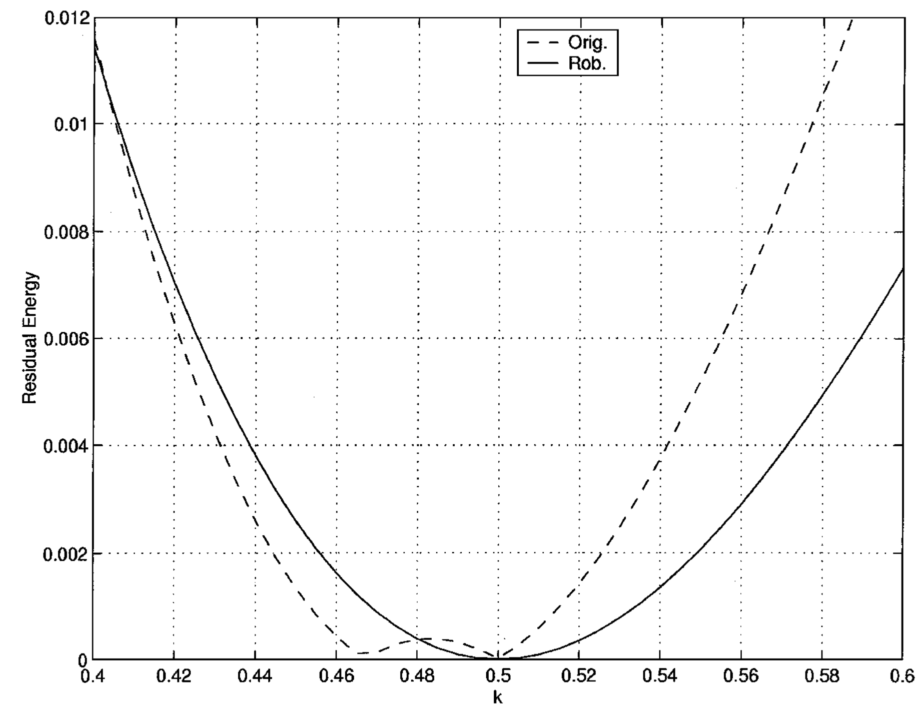

Fig. 11 Residual energy at final time vs *k* value for nonrobust and robust control (undamped).

system. Following the same procedure as just stated, the λ values for this system are

$$\lambda = \begin{bmatrix} 0.115933 \\ -0.023130 \\ 0.83E-6 \\ -0.011278 \\ 0.011957 \\ 0.005039 \\ -0.001404 \\ -0.000987 \end{bmatrix} \tag{55}$$

and the control is given as

$$\begin{aligned} u(t) &= 0.115933 - 0.023130t + (0.83E-6)e^{t} - (0.011278)e^{-t} \\ &\quad + 0.011957e^{(t/4)} \sin\left[\left(\sqrt{15}/4\right)t\right] + 0.005039e^{(t/4)} \\ &\quad \times \cos\left[\left(\sqrt{15}/4\right)t\right] - 0.001404e^{(t/4)}t \sin\left[\left(\sqrt{15}/4\right)t\right] \\ &\quad - 0.000987e^{(t/4)}t \cos\left[\left(\sqrt{15}/4\right)t\right] \end{aligned} \tag{56}$$

Figure 12 is a plot of the control (input) and position of m_1 (output) for both the nonrobust solution [Eq. (34)] and robust solution [Eq. (56)] for this damped system when there are no errors in the

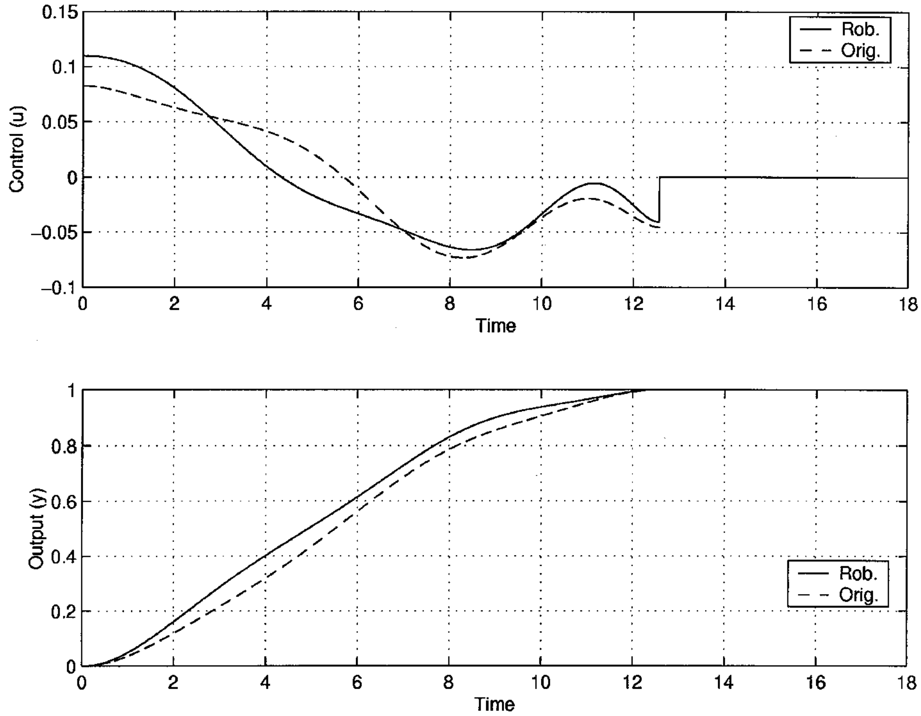

Fig. 12 Damped system nonrobust and robust control without error in *k*.

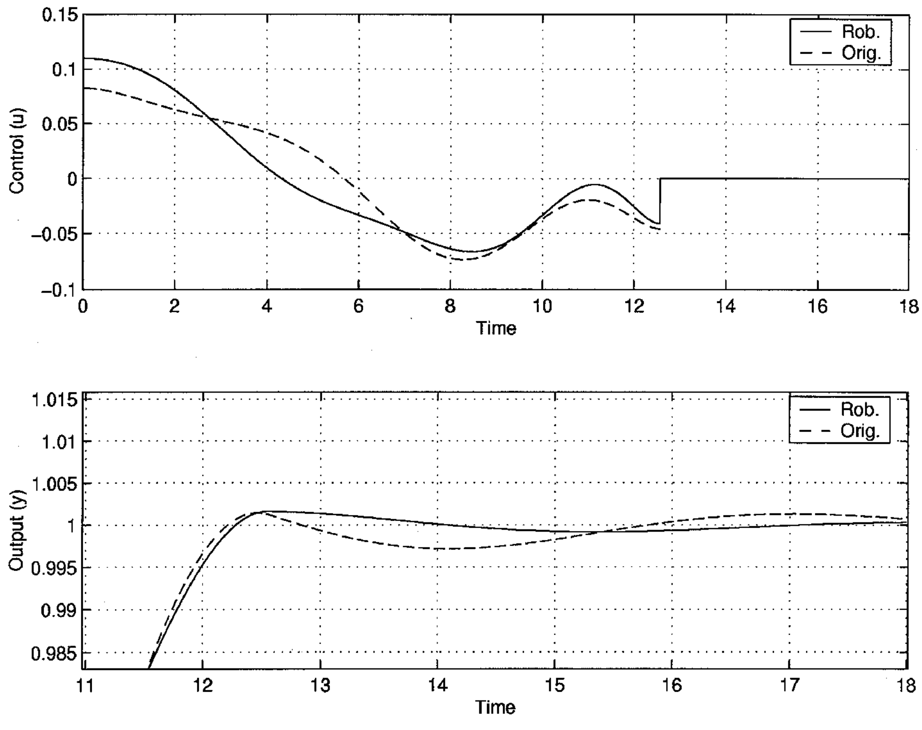

Fig. 13 Damped system nonrobust and robust control with 20% error in *k*.

system parameter k. Both solutions reach the final position without any residual vibration. Figure 13 is a plot of the same control input for the nonrobust and robust solutions shown in Fig. 12, this time when there is a 20% high error in k $[k = 1.2 * (\frac{1}{2})]$. The figure shows a reduction in the residual vibration caused by the error in the system parameter k with the robust solution.

Figure 14 is a plot of the residual energy in the system at the specified final time $(t_2 = 4\pi)$ vs the actual system k value using a design k value of $\frac{1}{2}$ for both the nonrobust and robust solutions. From this figure the robust solution for this damped system reduces residual vibration for all actual system k values shown in Fig. 14, with the exception that both the nonrobust and robust solutions will have zero residual vibration at the design value of $k = \frac{1}{2}$.

This section has demonstrated the benefit of the derived sensitivity equations when there are errors present in system parameters. Both the undamped and damped systems considered display reduced residual vibration with the addition of sensitivity equations. Though this technique may not reduce residual vibration for all actual system k values (as seen in Fig. 11), it does prove to be a useful method to locally reduce the residual vibration.

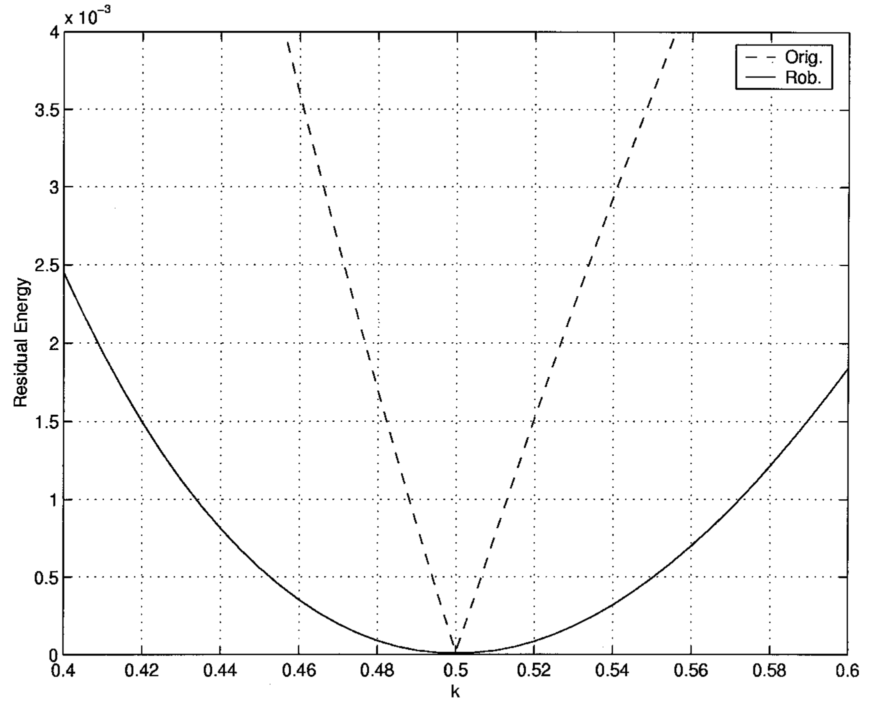

Fig. 14 Residual energy at final time vs k value for nonrobust and robust control (damped).

VI. Conclusions

A systematic procedure to obtain the closed-form solution for the rest-to-rest maneuver of the benchmark problem has been introduced, which minimizes the weighted power/jerk cost function. It has been shown that using the closed-form solution obtained for the minimum power/jerk control the minimum jerk and minimum power solutions are approached as the weighting parameter ζ approaches zero and infinity, respectively. The concept of sensitivity equations has been introduced, which, when added to the system state equations, gives a control that is robust to errors in system parameters. It has been shown that this robust control reduces residual vibration when the actual system parameter is in the vicinity of the design parameter used to derive the control. Though the robust solution obtained may not be an improvement for all possible errors in the system parameters, particularly when the parameter error is large, it does prove to be a useful technique for small perturbations thus giving a locally robust solution. Extensions of this work will include a study of the effect of varying damping as well as the application of this technique to more complicated systems.

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