Shaped Input Control of a System With Multiple Modes

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This paper describes a method for limiting vibration in flexible systems that have more than one characteristic frequency and mode. It is only necessary to have knowledge of the component mode frequencies and damping ratios in order to be able to calculate the timing and magnitudes of the impulse sequence used in the shaping. Only two impulses, in the nonrobust case, or three impulses in a more robust case, are necessary regardless of the number of component frequencies. Simple tests are established to determine when this technique can be used and examples are presented.

1 Introduction

There is a great deal of interest in finding control methods that will eliminate vibration from a wide variety of mechanical and structural systems. Examples range over satellites, robots, ships, aircraft and many more. One particular group of control strategies that has received recent attention is the group often referred to as shaped input control methods. These are largely based on the ideas introduced by O. J. M. Smith (1958) with his posicast control scheme where a step input was divided into two smaller steps that were delayed in time. This resulted in a reduction in the settling time of the system but it was not suitably robust for general application.

The robustness issue was successfully addressed by Singer and Seering (1988; 1989; 1990), Singhose et al. (1990), and by Hyde and Seering (1991a; 1991b) through the introduction of a third impulse to the impulse sequence. This third impulse arises from the introduction of the additional requirement that the system be insensitive to variations in the natural frequency and in the damping ratio.

The development of these methods is carried out for systems with one vibrational mode although Singer and Seering (1988; 1990) point out that multimode systems can be dealt with by convolving the impulse sequences for each individual mode with one another. Hence for a system with N modes, after the necessary convolutions, the control impulse sequence would contain 2^N impulses in the nonrobust case or 3^N impulses in the robust case. Clearly it would be most desirable to use the more robust three impulse control strategy, but it is also clear that the number of impulses will become very large for systems that require the inclusion of several modes in their model. Also, the higher modes will have higher frequencies, for example in the case of a cantilever beam the frequencies increase approximately quadratically with wave number (Dugundji, 1988) (i.e., $\omega_i \propto (i)^2$) and corresponding to these high frequencies are small periods. This means that the frequency of the impulses in the control impulse sequence may be very high and may even be outside of the bandwidth of the actuator being used with the result that the high frequency components may not be controllable by the method suggested in (Singer and Seering, 1988; Singer and Seering, 1990). In a later paper Hyde and Seering (1991a) deal explicitly with the multiple mode case and show that the requirement for 2^N or 3^N impulses can be reduced to $N + 1$ impulses or $2N + 1$ impulses, respectively.

Bhat and Miu (1990) have shown that by considering the problem in the Laplace domain it may be proved that a necessary and sufficient condition for eliminating residual vibration is that the finite time Laplace transform of the control input have a magnitude of zero at the system poles. This is achieved by finding an impulse sequence with the correct spacing and amplitude such that the Laplace transform will have zero magnitude in the right places. Solving for these amplitudes and time spacings can be very difficult for large numbers of modes and may even be very difficult for as few as two modes if the component mode frequencies are in an uncooperative ratio. Hyde and Seering (1991b) note that there are some modal spacings that prevent the exact solution of the spacing, and the amplitudes from being found. In the two mode case, for example, if the second mode is greater than three times the first, the method used to find the amplitudes and spacings will not converge to an exact solution (Hyde and Seering, 1991b).

Further to the issue of the number of impulses required is that there is a significant advantage in minimizing the number of impulses used to condition the input when the shaped input filter is used inside a feedback loop (Zuo and Wang, 1992).

In the remainder of the paper we will present a method that will allow a multimode system to be controlled by a shaped input technique with only two impulses in the nonrobust case or by three impulses in the robust case regardless of the number of modes present in the system. All that is required is knowledge of the modal frequencies and damping ratios. Thus an alternative to having to calculate a possibly large number of amplitudes and spacings for a multimode system is provided.

We begin with the undamped case in order to make the method clear and also in order to establish certain results that are important for both the damped and the undamped cases. Once these results are established the damped case is considered. Examples are used throughout to illustrate important points.

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2 Theoretical Development

Consider a function which may be expressed as a linear combination of sine functions each with a different frequency content. For example,

$$y(t)=\sum_{i=1}^{N} A_i \sin(\omega_i t) \tag{1}$$

If each of the component periods are related to all the other component periods by a rational constant then $y(t)$ will be periodic. This then means that if ω_i/ω_j is irrational (e.g., $=\sqrt{2}$) then $y(t)$ cannot be periodic. This problem is not likely to arise in practice because, for a real system, the frequencies will not be known to the precision implied by $\sqrt{2}$ but rather they will be known in the form 1.414, for example. The actual number of digits of accuracy depends on the method used to estimate the frequencies but what ever number it is it will be finite and the ratio of frequencies may be written as a rational. Then assuming that $y(t)$ is periodic it is necessary to find its period T such that, $y(t) = y(t+T)$. The period of $y(t)$ can be found by first considering the period of each individual term in (1) given by

$$T_i=\frac{2\pi}{\omega_i} \tag{2}$$

The period of $y(t)$ is given by the least common multiple (LCM) of the periods of the component modes such that

$$T=\mathrm{LCM}\{T_1, T_2, \ldots, T_N\} \tag{3}$$

This means that T may be expressed as some multiple of each of the component mode periods such that

$$k_i \in \{\text{natural numbers}\}$$

$$T=k_iT_i \quad i=1, \ldots, N$$

$$=2\pi\frac{k_i}{\omega_i} \tag{4}$$

Now, define ω_c as the frequency of the composite signal; ω_c may be expressed in terms of T in the usual manner. Hence,

$$\omega_c=\frac{2\pi}{T} \tag{5}$$

Each member of the set of component mode frequencies $\{\omega_i\}$ may be written as

$$\omega_i=m_i\omega_c \quad m_i \in \{\text{natural numbers}\} \tag{6}$$

Then using Eqs. (5) and (4) results in

$$\omega_i=m_i\frac{2\pi}{T}=\frac{m_i}{k_i}\frac{2\pi}{T_i} \tag{7}$$

which means that $m_i = k_i$.

In order for the posicast control method (Smith, 1958) or the shaped input control techniques (Singer and Seering, 1989; Singhose et al., 1990) to be applied it is necessary that the signal have what may be called a period of antisymmetry or an antiperiod T_a such that $y(t) = -y(t+T_a)$. For example the antiperiod for $\sin(t)$ or $\cos(t)$ is π.

In a similar manner to the above the antiperiod T_a will be given by the LCM of the component mode antiperiods such that

$$T_a=\mathrm{LCM}\{T_{a1}, T_{a2}, \ldots, T_{aN}\} \tag{8}$$

which in turn means that for any of the component terms of (1) that the value of T_a may be expressed as component mode half period plus some number v_i of component mode periods such that,

$$T_a=\frac{\pi}{\omega_i}+v_i\frac{2\pi}{\omega_i} \tag{9}$$

and consequently

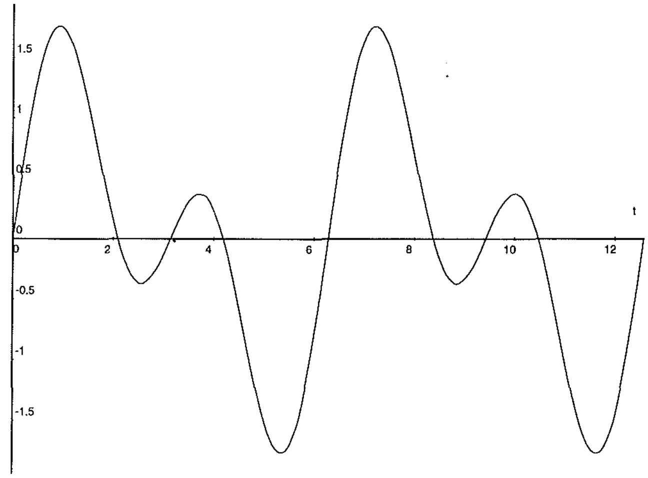

Fig. 1 $y_e(t) = \sin(t) + \sin(2t)$

$$\frac{\pi(2v_i+1)}{\omega_i}=\frac{\pi(2v_j+1)}{\omega_j} \; v_i,\, v_j \in \{\text{whole numbers}\}. \tag{10}$$

Also recall that $\omega_i = \omega_c m_i$ so that the above requirement may be cast as

$$\frac{2v_i+1}{m_i}=\frac{2v_j+1}{m_j} \; i\neq j \tag{11}$$

In order to find the values of v_i, it is necessary to satisfy $N(N-1)/2$ equations of the above form, however only $(N-1)$ of these equations are linearly independent. The above equations may also be expressed as

$$2m_jv_i-2m_iv_j=(m_i-m_j) \tag{12}$$

which is a linear Diophantine equation. The important point here is that a solution might not exist in which case the standard shaped input methods (Smith, 1958; Singer and Seering, 1989; Singhose et al., 1990) cannot be applied to the composite signal. A simple example of a signal $y_e(t)$ which has no antiperiod is shown in Fig. 1 where $y_e(t)$ is given according to,

$$y_e(t)=\sin(t)+\sin(2t) \tag{13}$$

Thus for a system with N modes there will be $N(N-1)/2$ simultaneous linear Diophantine equations in N unknowns to be solved in order to find T_a. Each of these equations is of the form (12).

The case of $N = 2$ is particularly simple, resulting in a single equation of the form

$$2m_2v_1-2m_1v_2=(m_1-m_2) \tag{14}$$

Note that because of the manner in which m_1 and m_2 are obtained they must be coprime. A single equation of this form has a solution iff $(m_1 - m_2)$ is divisible by the greatest common divisor (gcd) of $\{m_1, m_2\}$ and if a solution exists then an infinity of solutions exist (Gellert et al., 1989). Clearly if $(m_1 - m_2)$ is odd there can be no solution and if $(m_1 - m_2)$ is even there is an infinity of solutions. There are only two ways for $(m_1 - m_2)$ to be even; one is for m_1 and m_2 to be both odd, and the other is for them to be both even. But in order for m_1 and m_2 to be coprime they cannot be both even, hence the only possibility for a solution to Eq. (14) is for m_1 and m_2 to be both odd and coprime. The solution to Eq. (14) is

$$v_1=v_1'+m_1t \quad t\in\{\text{integers}\} \tag{15}$$

$$v_2=v_2'+m_2t \tag{16}$$

where v_1' and v_2' are the "special solving pair" found from the penultimate term of the continued fraction expansion of m_1/m_2 (Gellert et al., 1989).

It may be shown that a special solving pair for Eq. (14) is

$$v_1' = \frac{1}{2}(m_1 - 1) \quad v_2' = \frac{1}{2}(m_2 - 1) \tag{17}$$

and the general solution can be shown to be

$$v_1 = \frac{1}{2}(m_1 - 1) + m_1 t \quad v_2 = \frac{1}{2}(m_2 - 1) + m_2 t \tag{18}$$

For $v_i = 1/2(m_i - 1)$ $i = 1, 2$ (i.e., $t = 0$) $T_{a1} = T_{a2} = \pi/\omega_c$ and hence:

Result I: The minimum possible antiperiod of $y(t)$ is $T_a = \pi/\omega_c = 1/2T$.

Thus the antiperiod for $N = 2$, if it exists, is always half of the full period T. This result remains true for $N > 2$ as well, as may be demonstrated as follows.

Proof: By definition, T_a is the smallest T_a such that

$$y(t) = -y(t + T_a)$$

but

$$y(t + T_a) = -y(t + 2T_a)$$

therefore

$$y(t) = y(t + 2T_a)$$

and $T = 2T_a$. [Q.E.D.]

But as mentioned above there are $N(N-1)/2$ simultaneous linear Diophantine equations in N unknowns to be solved and the question of the existence of a solution in this case is more difficult than the discussion above which is only valid for $N = 2$. Because each of the equations has the form given in Eq. (12) it may be shown that only $(N - 1)$ of them are linearly independent. Hence if a solution exists there will be an infinity of solutions. In the case of more than two modes (i.e., $N > 2$) there is a solution $[v_1, v_2, \ldots, v_N]$ iff the gcd of the $(N - 1)$ rowed nonzero determinants of the coefficient matrix are equal to the gcd of the $(N - 1)$ rowed determinants of the augmented system (Gellert et al., 1989). Because of the specific form of this Diophantine system additional observations about the conditions that are necessary for a solution may be made. Notice that because of the manner in which they are constructed the m_i cannot be all even, at least one of them must be odd. But for any equation of the form (12) there will be a contradiction if $m_i - m_j$ is odd. Hence there can be no solution to our system of Diophantine equations unless all the m_i are odd (this is necessary but not sufficient).

As an example of how this may be used consider the case where

$$y_e(t) = \sin(30t) + \sin(105t) + \sin(385t) \tag{19}$$

such that $\omega_1 = 30$, $\omega_2 = 105$, $\omega_3 = 385$ and the period of $y_e(t)$ is $T = 2\pi/5$, the frequency is $\omega_c = 5$ and $m_1 = 6$, $m_2 = 21$, and $m_3 = 77$.

It can be concluded at this point that because $m_1 = 6$ is even there is no solution to the Diophantine system and hence no *antiperiod* but in order to illustrate the broader method consider the following. In order to determine whether or not an antiperiod exists it is necessary to examine the following system in accordance with the above. There are three equations in three unknowns given by

$$\begin{bmatrix} 2m_2 & -2m_1 & 0 \\ 2m_3 & 0 & -2m_1 \\ 0 & 2m_3 & -2m_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} m_1 - m_2 \\ m_1 - m_3 \\ m_2 - m_3 \end{bmatrix} \tag{20}$$

An examination of the coefficient matrix on the left-hand side will show that it has rank = 2. Considering, for example, the determinants of the coefficient matrix based on the first two rows (it would not matter which two rows were picked) it may be shown that their gcd = 24 and in contrast the determinants of the augmented system have a gcd = 12. Hence by the criterion given above there is no solution to this system of

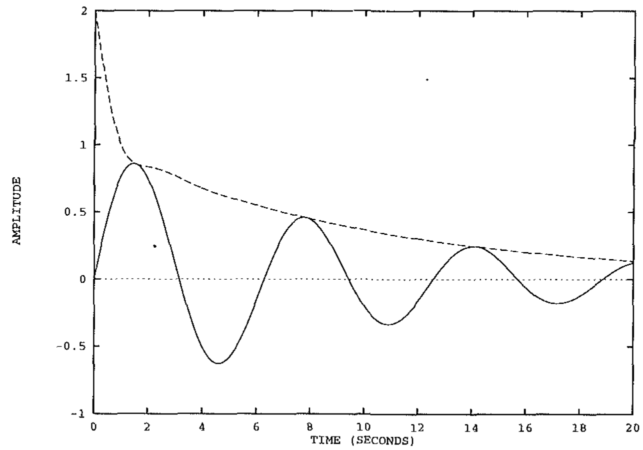

Fig. 2 Envelope and signal for $y_e(t) = \exp(-0.1t)\sin(t) + 1/3 \exp(-1.2t)\sin(3t)$

linear Diophantine equations and consequently there is no antisymmetry in $y_e(t)$. It is useful to note that although there is no antisymmetry present in $y_e(t)$ a subset of the modes may be combined into a signal which does possess antisymmetry. Specifically modes two and three may be comnbined into a signal which has antisymmetry. Thus $y_e(t)$ may be represented as the sum of two periodic functions rather than three which will permit the use of fewer impulses in the control sequence than if a mode by mode approach were taken.

The foregoing allows us to apply the standard shaped input techniques to signals of the form given in Eq. (1). But signals of this form are undamped and of less practical interest than damped signals of the form

$$y(t) = \sum_{i=1}^{N} A_i \exp\left(-\frac{t}{\tau_i}\right) \sin(\omega_{di} t) \tag{21}$$

The individual damped natural frequencies ω_{di} can be re-expressed in a manner similar to the way the frequencies were expressed in the undamped case such that

$$\omega_{di} = m_i \omega_{dc} \tag{22}$$

and hence

$$y(t) = \sum_{i=1}^{N} A_i \exp\left(-\frac{t}{\tau_i}\right) \sin(m_i \omega_{dc} t) \tag{23}$$

In order to proceed the following result from trigonometry is needed.

$$\sin(m_i \omega_{dc} t) = \sum_{r=1,3,5,\ldots}^{m_i \geq r} (-1)^{r-1/2} \, {}_{m_i}\mathcal{B}_r \sin^r(\omega_{dc} t) \cos^{m_i - r}(\omega_{dc} t) \tag{24}$$

where ${}_{m_i}\mathcal{B}_r$ is the binomial coefficient. The proper interpretation of the upper limit of the summation is that r is to be increased so long as it does not exceed m_i. Then using Eq. (24) in Eq. (23) it can be shown that

$$y(t) = \sin(\omega_{dc} t) E(t) \tag{25}$$

where after a change of indices such that $r = (2j - 1)$,

$$E(t) = \left[\sum_{i=1}^{N} \sum_{j=1}^{m_i \geq 2j-1} A_i \exp\left(-\frac{t}{\tau_i}\right) (-1)^{j-1} \, {}_{m_i}\mathcal{B}_{2j-1} \times \sin^{2j-2}(\omega_{dc} t) \cos^{m_i - 2j + 1}(\omega_{dc} t) \right] \tag{26}$$

is the envelope of the system response. It is the generalization of the exponential envelope of the single degree of freedom case. This envelope function is essential to being able to apply the shaped input techniques to the overall signal $y(t)$.

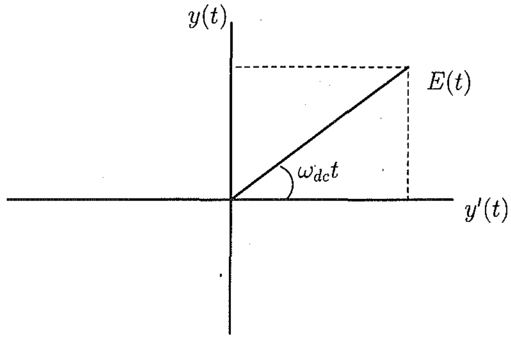

Fig. 3 Impulse response, illustrated by means of a rotating vector

An example of this envelope is given in Fig. 2 where the signal $y_e(t)$ is given by

$$y_e(t) = \exp(-0.1t)\sin(t) + \frac{1}{3}\exp(-1.2t)\sin(3t)$$

where it may be seen that the envelope function $E(t)$ behaves in a similar manner to the single degree of freedom case.

3 The Shaped Input Method

The response of a system with multiple frequencies that has anti-symmetry can be rewritten in the form given in Eq. (25). It is useful to rewrite the envelope function $E(t)$ in the form

$$E(t) = \sum_{i=1}^{N} \sum_{j=1}^{m_i \geq 2j-1} A_i \exp(-f(\xi_i) m_i \omega_{dc} t)(-1)^{j+1}{}_{m_i}\mathcal{B}_{2j-1} \times \sin^{2j-2}(\omega_{dc}t)\cos^{m_i-2j+1}(\omega_{dc}t) \quad (27)$$

where $f(\xi_i) = \xi_i/\sqrt{1-\xi_i^2}$. The above equation represents the impulse response of a system. This can be illustrated by means of a vector, rotating at a frequency of ω_{dc} radians/s and with an amplitude of $E(t)$. The projection of this rotating vector on the y-axis at any time is the response of the system (Fig. 3). The projection of the rotating vector on the x-axis is

$$y'(t) = \cos(\omega_{dc}t)E(t) \quad (28)$$

The response of the system to a sequence of k impulses can be expressed for time $t > t_k$ as

$$y(t) = \sum_k I_k \sin(\omega_{dc}(t-t_k))E(t-t_k) \quad (29)$$

where I_k and t_k are the amplitude and time of application of the kth impulse.

3.1 Two Impulse Controller. For a two impulse response, assume $I_2 = 1$, $t_1 = 0$ and $t = t_2$. From the phasor diagram (Fig. 4), the addition of the components of the vectors in the vertical and horizontal directions gives us

$$I_1\cos(\omega_{dc}t_2)E(t_2) + \left[\sum_{i=1}^{N} A_i m_i\right] = 0 \quad (30)$$

and

$$I_1\sin(\omega_{dc}t_2)E(t_2) = 0 \quad (31)$$

From Eq. (31) it may be found that

$$t_2 = \frac{\pi}{\omega_{dc}} \quad (32)$$

Using this value of t_2 in Eq. (30) gives

$$-I_1\left[\sum_{i=1}^{N} A_i \exp(-f(\xi_i)m_i\pi)m_i(-1)^{m_i-1}\right] + \left[\sum_{i=1}^{N} A_i m_i\right] = 0 \quad (33)$$

which can also be written as

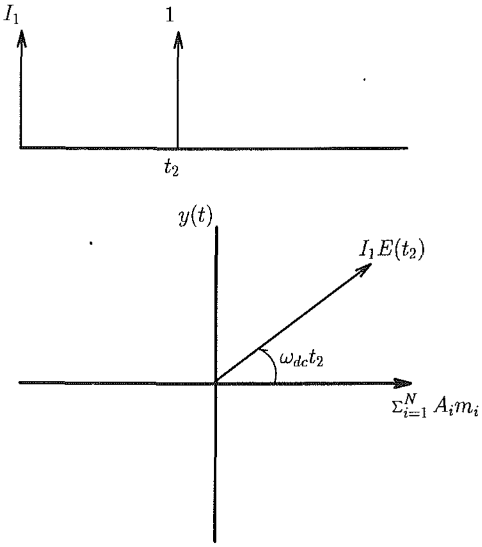

Fig. 4 Phasor diagram for a two impulse controller

$$I_1 = \frac{\sum_{i=1}^{N} A_i m_i}{\sum_{i=1}^{N} A_i \exp(-f(\xi_i)m_i\pi)m_i(-1)^{m_i-1}} \quad (34)$$

With knowledge that the m_i must be odd the impulse sequence for the multi-frequency system is given in the following table.

time	amplitude
0	$\dfrac{\sum_{i=1}^{N} A_i m_i}{\sum_{i=1}^{N} A_i \exp(-f(\xi_i)m_i\pi)m_i}$
$\dfrac{\pi}{\omega_{dc}}$	1

3.2 Three Impulse Controller. The two impulse, shaped input controller is known to be sensitive to errors in estimated frequencies. To overcome this problem the use of three impulses has been suggested by Seering and Singer (1988). The three impulse controller for the multi-frequency system can be developed in the same fashion as the two impulse controller. Assuming that $I_3 = 1$, $t_1 = 0$ and $t = t_3$, the phasor diagram (Fig. 5) yields

$$I_1\cos(\omega_{dc}t_3)E(t_3) + I_2\cos(\omega_{dc}(t_3-t_2))E(t_3-t_2) + \left[\sum_{i=1}^{N} A_i m_i\right] = 0 \quad (35)$$

and

$$I_1\sin(\omega_{dc}t_3)E(t_3) + I_2\sin(\omega_{dc}(t_3-t_2))E(t_3-t_2) = 0 \quad (36)$$

There are now two equations in four unknowns (t_3, t_3, I_1, and I_2). Two additional equations are obtained by differentiating equations of the form of Eq. (29) and Eq. (28) with respect to ω_{dc} which corresponds to minimizing the error due to errors in estimated frequency. That is we require

$$\frac{d\sum_k I_k \sin(\omega_{dc}(t-t_k))E(t-t_k)}{d\omega_{dc}} = 0 \quad (37)$$

and

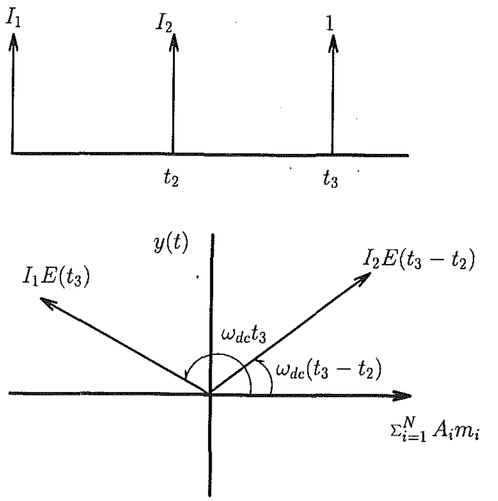

Fig. 5 Phasor diagram for a three impulse controller

Table 1 Impulse sequences for the undamped problem

Two impulse case		Three impulse case	
time	amplitude	time	amplitude
0	0.5	0	0.25
$\pi/2$	0.5	$\pi/2$	0.5
		π	0.25

$$\frac{d\sum_k I_k\cos(\omega_{dc}(t-t_k))E(t-t_k)}{d\omega_{dc}}=0 \tag{38}$$

Solving Eqs. (35), (36), (37), and (38) produces the amplitudes of the impulses I_1 and I_2 and the time of application t_2 and t_3. An iterative numerical solution scheme is used in the solution process.

4 Example

4.1 An Undamped System. Assume that the system response to an impulse input can be represented in the form

$$y(t)=\sin(\omega_1 t)+\sin(\omega_2 t) \tag{39}$$

where $\omega_1 = 2$ and $\omega_2 = 14$ and hence $\omega_c = 2$, $m_1 = 1$ and $m_2 = 7$.

The integers m_1 and m_2 are coprime and are both odd thus guaranteeing that antisymmetry exists. The common period for the system is $T = 2\pi/3 = \pi$ and the period of antisymmetry is $T_a = \pi/2$.

Using the two impulse controller results from Eqs. (32) and (34) on the above system yields $t_2 = \pi/2$ and $I_1 = 1$. Then scaling the impulses so that the sum of their magnitudes is unity results in the impulse sequence given in Table 1.

Equation (39) is plotted in Fig. 6 where it may be seen that antisymmetry exists in this signal. Using a sampling interval of a hundredth of a second the simulation of the two impulse controller (Figs. 7 and 8) shows a fairly good attenuation of the amplitudes of vibration. The response of the system, when the second impulse is applied one sampling interval after the desired time, results in the frequencies not being totally eliminated. The residual vibration confirms the lack of robustness of the two impulse controller.

The amplitudes of the impulses for the three impulse sequence are arrived at from Eq. (35), (36), (37), and (38). Scaling the amplitudes similar to the two impulse case results in the values given in Table 1.

The simulated results show a marked improvement in the elimination of the residual vibration even when the error in the time of application of the second and third impulse is one sampling interval (Figures 9 and 10).

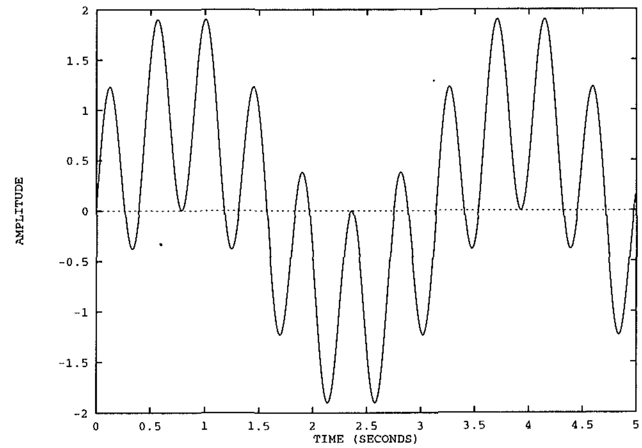

Fig. 6 Response of a system without damping to a unit impulse

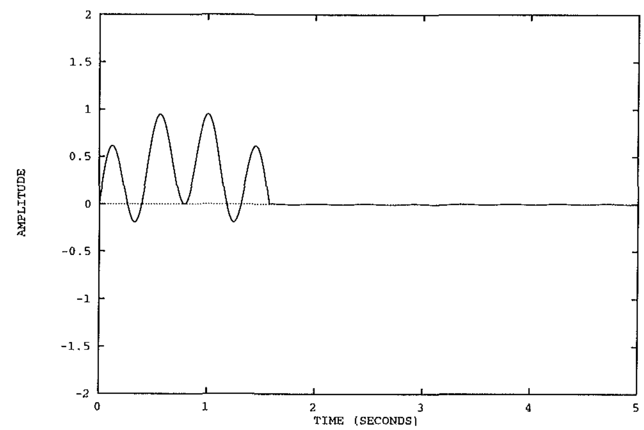

Fig. 7 Response of a system without damping to a two impulse controller; perfect application of the second impulse

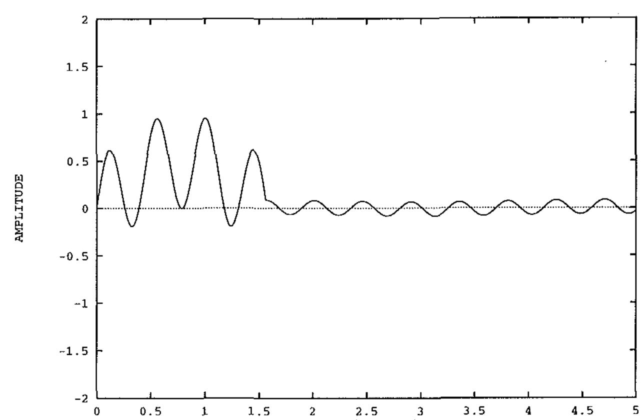

Fig. 8 Response of a system without damping to a two impulse controller; residual vibration due to imperfect application of the second impulse

4.2 An Under-Damped System. For the purposes of illustration assume that the system response to an impulse input can be represented in the form

$$y(t)=\exp(-\xi_1\omega_1 t)\sin(\omega_1 t)+\exp(-\xi_2\omega_2 t)\sin(\omega_2 t) \tag{40}$$

where $\xi_1 = \xi_2 = 0.01$ and the values of ω_1 and ω_2 are the same as in the previous example.

For the case of the two impulse controller acting on the

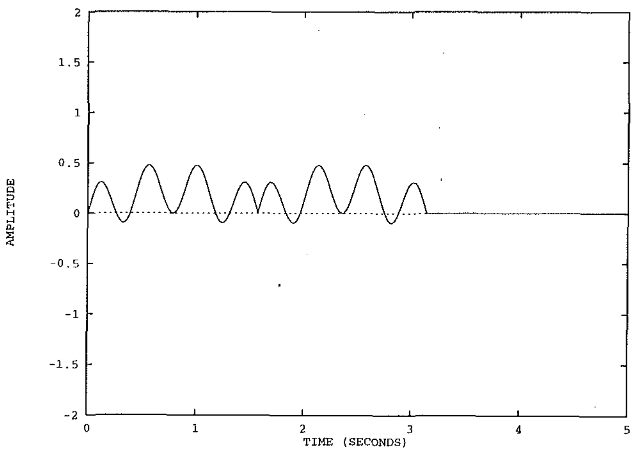

Fig. 9 Response of a system without damping to a three impulse controller; perfect application of the third impulse

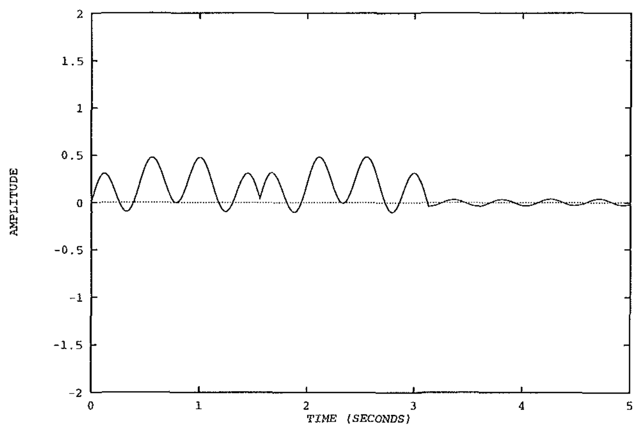

Fig. 10 Response of a system without damping to a three impulse controller; residual vibration due to imperfect application of the third impulse

Table 2 Impulse sequences for the under-damped problem

Two impulse case		Three impulse case	
time	amplitude	time	amplitude
0	0.5484	0	0.30644
$\pi/2$	0.4516	1.57	0.49482
		3.14	0.19874

above damped system, Eqs. (32) and (34) yield $t_2 = \pi/2$ and $I_1 = 1.2144$. Scaling the impulses so that the sum of their magnitudes is unity produces the impulse sequence given in Table 2.

For the purposes of comparison the response of the system to the classical shaped input method where four impulses were used is shown in Fig. 11. Simulation of the system response to the two impulse control sequence, which is based on the above developments (Fig. 12) with a sampling interval of a hundredth of a second shows that the frequencies are not totally eliminated but are reduced compared to the response of the system to a unit impulse (Fig. 13). The residual vibration in the two impulse case is larger than that in the four impulse case due to the fact that the two impulse controller is trying to do in two impulses what is being done by four impulses in the former case. In both examples the last impulse was not applied precisely at the exact temporal instant desired because of the frequency content and the choice of step size. This indicates that the two impulse case lacks robustness, as expected, but also indicates that the classical shaped input method with 2^N impulses may be more robust in the multimode case due simply to the presence of more impulses.

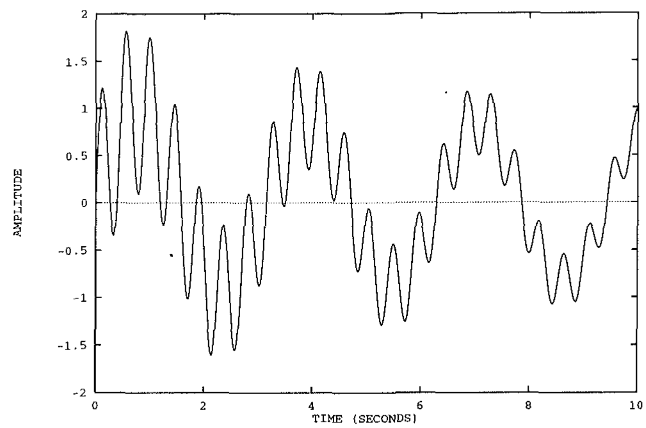

Fig. 11 Response of a system with damping to a two impulse controller

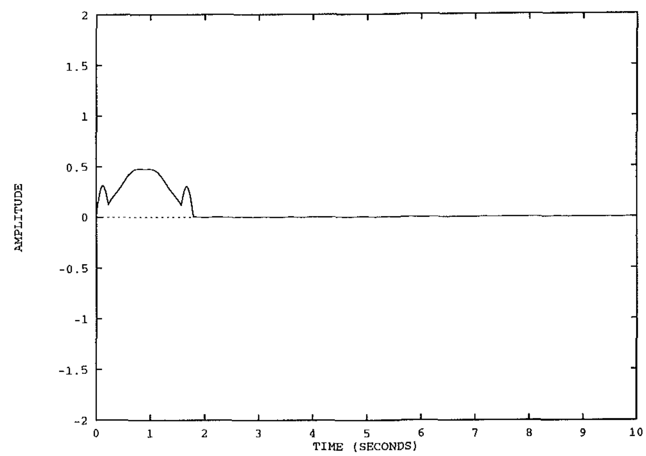

Fig. 12 Response of a system with damping to a unit impulse

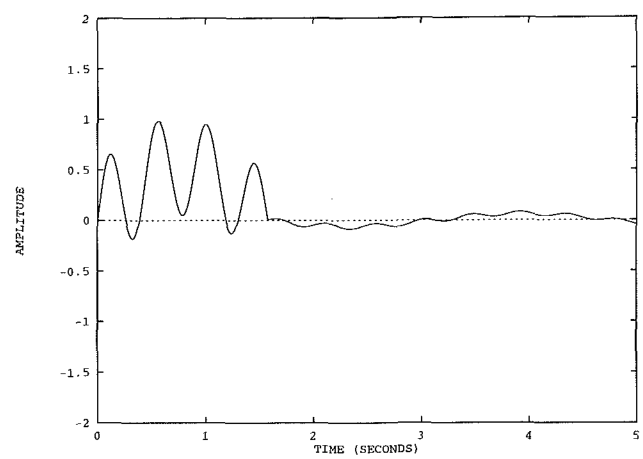

Fig. 13 Response of a system with damping to a classical shaped input controller with four impulses

The amplitudes and timing of the impulses for the three impulse sequence are arrived at from Eqs. (35), (35), (37), and (38). Scaling the amplitudes similar to the two impulse case results in the impulse amplitudes given in Table 2. The simulated response of the system to a classical shaped input control impulse sequence consisting of nine (3^N) impulses is illustrated in Fig. 14. This response can be compared with the response given in Fig. 15 which illustrates the response of the system to the three impulse controller with the above impulse mag-

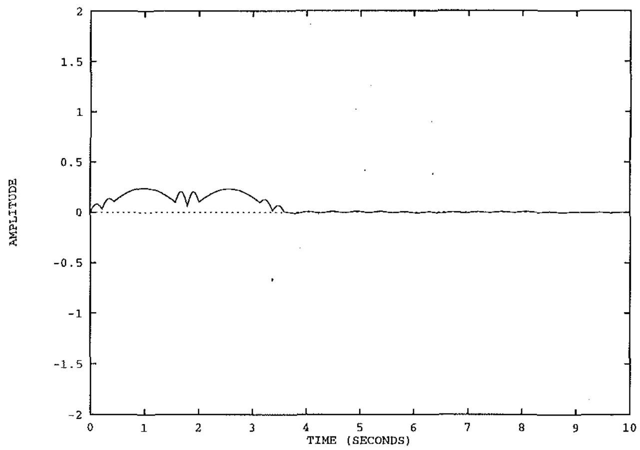

Fig. 14 Response of a system with damping to a classical shaped input controller with nine impulses

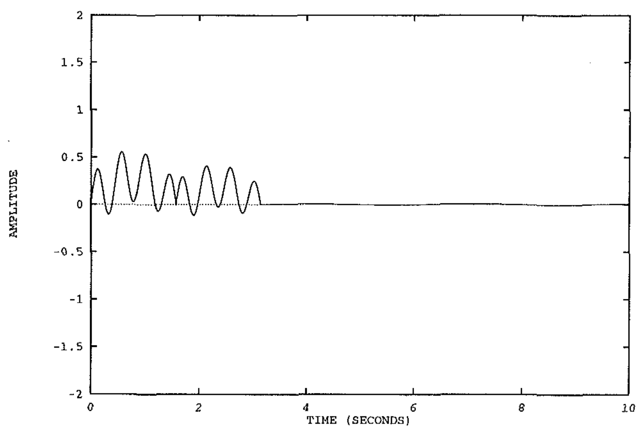

Fig. 15 Response of a system with damping to a three impulse controller

nitudes. The simulated results show that the three impulse controller eliminates the vibrations just as effectively as the nine impulse controller and is more robust than the two impulse case, as expected. It may be noted that in the case of the three impulse controller, the system vibrations, prior to the end of the impulse sequence, are not diminished as much as in the nine impulse sequence case. This is because fewer impulses are applied during that time and each impulse that is applied is necessarily larger in magnitude that in the nine impulse case.

5 Summary and Conclusions

A technique for the design of a shaped input controller for the suppression of vibration of systems with multiple modes has been presented. The suppression is achieved by designing an impulse sequence which cancels a pseudo-mode of lower frequency than any of the component modes whose frequencies must necessarily be greater than or equal to that of the pseudo-mode. The impulse train so designed cancels all the higher frequency component modes and thus eliminates the vibration from the system.

A minimum of two impulses are required to eliminate all the modes of vibration provided antisymmetry exists in the response. The robustness issue has been addressed leading to a three or more impulse sequence for the elimination of the vibration. By being able to use three impulses to cancel the vibration the method presented above may eliminate the need to find a large number of impulse magnitudes and spacings; a task which in practice may be very difficult. Computer simulations have corroborated the applicability of this technique and have also confirmed the lack of robustness of the two impulse sequence controller. An important point to note is that the time required for the elimination of all the modes will be greater than or equal to the period of the lowest natural frequency. In this regard it may be observed that if the component frequencies are known to a large number of digits of accuracy this will have the effect of lengthening the period of the total signal. This effect can be diminished by using a continued fraction expansion on the decimal forms of the frequencies to get as compact a form as possible. This may introduce small errors into the values of the component frequencies but the robustness of the three impulse scheme should be able to accommodate this.

A note to the use of this technique: Even when antisymmetry cannot be found for the system including all the modes, we are still at liberty to use subsets of the frequency spectrum and design impulse sequences for those subsets that provide antisymmetry and convolve the impulse sequences for each of the subsets, which would still reduce the number of impulses required as compared to convolving impulse sequences for all the frequencies repeatedly.

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