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Robust time-delay control of multimode systems

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This paper presents a procedure for the design of open loop controllers for flexible structures using multiple step inputs delayed in time. The controller attenuates the residual vibration by cancelling the complex poles of the system. Robustness is achieved by locating additional zeros at the cancelled poles of the system. The paper begins by addressing the control of a single mode and examines the effect of user selected time-delays on robustness and the reference input. Next, a procedure for the design of robust time-delay controllers for multiple modes with user selected time-delays is considered. This is followed by a design of a minimum time-delay controller, such that the step input magnitudes are constrained to values between 0 and 1. Two examples, a spring–mass system and a single-link flexible-arm robot are used to illustrate the effectiveness of the proposed controller.

1. Introduction

With the current interest in space stations, retargeting space structures, and space-based robots, there has been a number of studies on the control of vibration induced by rapid reorientation of the structure itself or its subsystems. Considerable research effort has gone into the design of closed and open-loop control strategies. A fairly comprehensive treatment of this family of problems has been presented by Junkins and Turner (1986).

Rapid manoeuvring of a flexible structure between two quiescent attitudes with the objective of minimizing structural vibration has been addressed by many researchers (Farrenkopf 1979, Junkins *et al.* 1990, Vadali *et al.* 1992). Farrenkopf (1979) arrived at control profiles that minimize a cost criterion—a function of residual modal energy and the integral of the modal velocity and acceleration. The optimal control profiles were generated for a spacecraft with one flexible appendage containing one mode of vibration. Swigert (1980) used a different cost function, one which reflects the sensitivity of the terminal modal amplitudes to variations in modal frequencies. He proposed a control profile that is a sum of multiple harmonic functions and showed that the control profile used to minimize torque is very sensitive to parameter errors. The cost function used by Singh *et al.* (1989) was the terminal time. The optimal control profiles for time-optimal, rest-to-rest slewing manoeuvres of flexible spacecraft were shown to be bang-bang with multiple switches in each control variable. Hablani (1990) developed a technique for the estimation of the switch times for a bang-bang profile for the zero-residual energy, single-axis slew of a flexible spacecraft. Thompson *et al.* (1989) proposed a near-minimum-time control profile which involved approximating the signum function by a hyperbolic

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tangent function. Junkins *et al.* (1990) used a piecewise continuous spline approximation to the signum function such that the boundary conditions are met. This smooth profile does not 'ring' the higher modes of the structure. All the above references considered single-axis manoeuvres. Vadali *et al.* (1992) synthesized control profiles for near minimum time manoeuvres of space structures using parameter optimization. They used the Sequential Quadratic Programming method to arrive at the optimum switch times and maximum thrust amplitudes. Of these works, only Swigert (1980) considered robustness issues, such as sensitivity to errors in natural frequencies.

We wish to place special emphasis on the following works in view of their relevance to the controller proposed in this paper. A very interesting technique was proposed by Tallman and Smith (1958) which involved splitting the input excitation into several segments such that the sum of all transient terms equals zero after the last excitation. This technique was referred to as the *posicast technique*. This work acknowledged the lack of robustness to errors in estimated damping and frequency of the controlled system. This controller can also be represented as a proportional and time-delayed block. Singer and Seering (1990) proposed a technique for decreasing the sensitivity of the posicast controller to modelling errors, which they referred to as the *shaped input* controller. The design of the controller involved studying the response of the system to a sequence of impulses. The amplitudes and times of application of the impulses were determined by solving a set of equations representing the response of the system to the impulse sequence and, for robustness, derivatives of these equations with respect to frequency or damping. Singhose *et al.* (1990) used the phasor diagram to arrive at the same solution. The input shaping technique discussed above is designed to suppress residual vibration of one mode. To eliminate residual vibration due to multiple modes, the impulse sequence for each mode is convolved to produce a new impulse sequence. Wie and Liu (1990) used the shaped input technique to modify the flexible-body, time optimal control profile to produce a robust control scheme that was applied to the control of a two-mass spring problem. Murphy and Watanabe (1992) showed that the shaping filter zeros in the discrete domain, exactly cancel the plant poles, leading to elimination of dynamics corresponding to the cancelled poles. Hyde and Seering (1991) developed a numerical technique for arriving at an impulse train for the control of multiple mode systems, which is simple to implement and produces a smaller system response delay as compared with using cascades of shaped input controllers designed for each mode separately. This technique involved the direct solution of a group of simultaneous nonlinear impulse constraint equations. Singh and Heppler (1993) arrived at a method for suppressing vibration for systems with multiple modes, provided certain constraints related to the anti-symmetric periodic property of the signal are satisfied. Two impulses for the non-robust case and three or more impulses for the more robust case are required for this controller.

Time-delay control has been used in feedback form in papers by Suh and Bien (1979), Youcef-Toumi and Kondo (1989) and in open loop form by Singh and Vadali (1993). Suh and Bien (1979) proposed a controller which utilized the time-delay effect in conjunction with a proportional element. This controller was referred to as proportional minus delay (PMD) controller. This controller was shown to be insensitive to high-frequency noise and has a smaller 'Integral of the Time weighted Absolute Error (ITAE)', than a PD controller. Swisher and Tenqchen (1988) extended this work to a third-order system and arrived at rules of thumb for the selection of the parameters of the PMD controller. Su *et al.* (1989) used time-delay for the tip control of a flexible beam. Their work used time-delay in minor loops attached to the main loop and showed the controller as an infinite-dimensional and complex generalization of the classical lead–lag compensator. Youcef-Toumi and Kondo (1989) used the time-delay controller for a system whose control distribution matrix is unknown.

In this paper, we first design a time-delay controller for a single mode system and show that multiple uses of this controller in cascade lead to controllers that are insensitive to parameter errors. We then design a time-delay controller, where the time-delay is user selected, this is a generalization of the time-delay controller proposed by Singh and Vadali (1993). This is followed by studying some properties of the controller regarding the sign of the gains of the time-delay elements and periodicity of the controller. We refer to the magnitude of the delayed step input, as the time-delay gain. A general procedure for the design of user selected time-delay controllers for multi-mode systems is presented. The final controller presented in this paper is based on minimizing the time-delay with the gains of the time-delay elements constrained to lie between 0 and 1. A spring–mass system and a single-link flexible-arm robot are used to illustrate the effectiveness of the time-delay controller for single and multimode systems, respectively.

2. Time-delay control

2.1. *Proportional plus delay (PPD) control*

Time-delay controllers designed to cancel the poles of a system with the intention of attenuating the residual vibration have been shown to correspond to the two-impulse shaped-input controller (Singer and Seering 1990) by Singh and Vadali (1993). It has further been shown that the three-impulse shaped-input controller (designed to increase robustness) is equivalent to using two time-delay controllers in series. The design of the time-delay controller to cancel the poles of the system is presented in this section.

Figure 1 is used to represent the time-delay control of a second-order underdamped system. We need to determine A_0 and T so that the poles of the system are cancelled by the zeros of the controller, which are given by the equation

$$A_0 + \exp(-sT) = 0 \tag{1}$$

where we have normalized the relative amplitudes of the proportional and time-delayed signal. A_0 is the amplitude of the proportional signal and T is the delay time of the time-delayed signal.

Representing the Laplace variable s as

$$s = \sigma + j\omega \tag{2}$$

and substituting (2) into (1) and equating the real and imaginary parts to zero, we have

$$A_0 = \exp(-\sigma T)\cos(\omega T) = 0 \tag{3}$$

and

$$\exp(-\sigma T)\sin(\omega T) = 0 \tag{4}$$

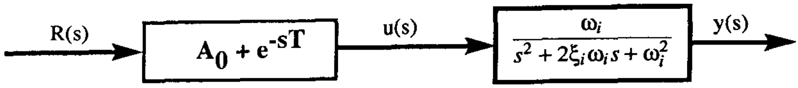

Figure 1. Single time-delay controlled system.

From (4) we have

$$\omega = (2n + 1)\frac{\pi}{T}, \quad 2n\frac{\pi}{T} \tag{5}$$

Substituting (5) into (3), we see that only $\omega = (2n + 1)\pi/T$ produces a positive value of A_0, which leads to

$$\sigma = -\frac{1}{T}\ln(A_0) \tag{6}$$

The zeros of the controller are

$$s = \begin{cases} \dfrac{-\ln(A_0) + (2n + 1)\pi j}{T} & A_0 > 0 \\ \dfrac{-\ln(-A_0) + 2n\pi j}{T} & A_0 < 0 \end{cases} \qquad n = -\infty, \ldots, 0, \ldots, \infty \tag{7}$$

In this work, we assume A_0 is positive and use the zeros corresponding to that assumption.

To cancel the system poles at $s = -\xi_i\omega_i \pm j\omega_i(1 - \xi_i^2)^{1/2}$, we have from (5) (setting $n = 0$)

$$\omega = \omega_i(1 - \xi_i^2)^{1/2} = \frac{\pi}{T} \tag{8}$$

$$\Rightarrow T = \frac{\pi}{\omega_i(1 - \xi_i^2)^{1/2}} \tag{9}$$

and

$$\sigma = -\xi_i\omega_i \tag{10}$$

Substituting (9) into (6), we have

$$-\ln(A_0) = -\xi_i\omega_i \frac{\pi}{\omega_i(1 - \xi_i^2)^{1/2}} \tag{11}$$

which leads to

$$A_0 = \exp\left[\frac{\xi_i\pi}{(1 - \xi_i^2)^{1/2}}\right] \tag{12}$$

This corresponds exactly to the solution of the *shaped-input* technique. The controller can also be written as

$$\begin{aligned} u(s) = (s^2 + 2\xi_i\omega_i s + \omega_i^2)&(s^2 + 2\xi_i\omega_i s + 9\omega_i^2 - 8\xi_i^2\omega_i^2) \\ \ldots (s^2 + 2\xi_i\omega_i s &+ n^2\omega_i^2 - (n^2 - 1)\xi_i^2\omega_i^2)R(s) \\ n &= 1, 3, 5, \ldots \end{aligned} \tag{13}$$

The pole–zero locations of the controlled system are shown in Fig. 2. Thus, the Single Time-Delay controller can also be used to cancel poles of the system that are odd multiples of the two primary poles.

The final value of the single time-delayed controlled system to a unit step input, is given by

$$\lim_{s\to 0}\frac{1}{s}s\left(\frac{A_0+\exp(-sT))\omega_i^2}{s^2+2\xi_i\omega_i s+\omega_i^2}\right)=A_0+1 \tag{14}$$

To force the final value of the input after passing through the controller be the same as that entering the controller, we normalize the amplitudes of the direct and time delayed signal, so that the time-delay controller transfer function is $(A_0+\exp(-sT))/(A_0+1)$.

2.2. *Proportional plus multiple delay (PPMD) control*

The Single Time-Delay control, by cancelling the poles corresponding to the oscillatory behaviour of the system, provides us with a technique to produce non-oscillatory response. The cancellation of the poles of the system is contingent on the availability of accurate data regarding them. To ameliorate the robustness of the Time-Delay control to errors in estimated poles, the two-Time-Delay controller is proposed. The controlled system is illustrated in Fig. 3. The zero of the controller is given by the equation

$$A_0+A_1\exp(-sT)+\exp(-2sT)=0 \tag{15}$$

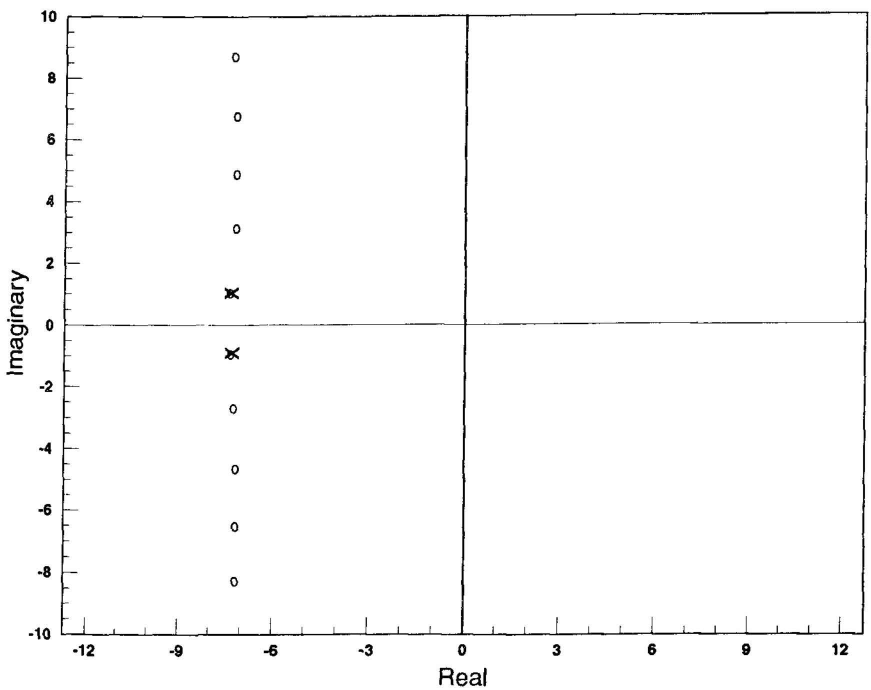

Figure 2. Pole–zero locations of the controlled system in the s-plane.

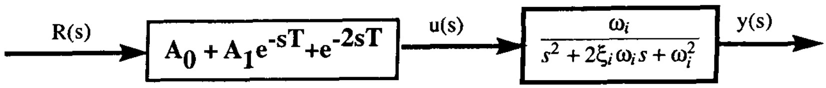

Figure 3. Two time-delay controlled system.

The tacit assumption made is that the second delay is twice the first delay. Letting $s = \sigma + j\omega$ and equating the real and imaginary parts to zero, we have

$$A_0 + A_1 \exp(-\sigma T)\cos(\omega T) + \exp -2\sigma T)\cos(2\omega T) = 0 \tag{16}$$

$$A_1 \exp(-\sigma T)\sin(\omega T) + \exp(-2\sigma T)\sin(2\omega T) = 0 \tag{17}$$

respectively. Equation (17) can be rewritten as

$$\sin(\omega T)\{A_1 \exp(-\sigma T) + 2\exp(-2\sigma T)\cos(\omega T)\} = 0 \tag{18}$$

the solution of which is

$$\omega = (2n + 1)\frac{\pi}{T}, \quad 2n\frac{\pi}{T} \tag{19}$$

or

$$\omega = \frac{1}{T}\cos^{-1}\left(-\frac{A_1}{2}\exp(\sigma T)\right) \tag{20}$$

Substituting $\omega = (2n + 1)\pi/T$ into (16), we have

$$A_0 - A_1 \exp(-\sigma T) + \exp(-2\sigma T) = 0 \tag{21}$$

To design a controller that is robust to errors in estimated frequencies, we equate the derivative of (21) with respect to σ to zero.

$$A_1 \exp(-\sigma T) - 2\exp(-2\sigma T) = 0 \tag{22}$$

We arrive at the same equation if we differentiate (18) with respect to ω and equate it to zero. Solving (22), we have

$$A_1 = 2\exp(-\sigma T) \tag{23}$$

Substituting (23) into (21), we have

$$A_0 = \exp(-2\sigma T) \tag{24}$$

To cancel the system poles, we equate

$$\sigma = -\xi_i \omega_i \tag{25}$$

and

$$\omega = \omega_i(1 - \xi_i^2)^{1/2} = \frac{\pi}{T} \tag{26}$$

Thus we have

$$A_0 = \exp\left[\frac{2\pi\xi_i}{(1 - \xi_i^2)^{1/2}}\right] \tag{27}$$

$$A_1 = 2\exp\left[\frac{\pi\xi_i}{(1 - \xi_i^2)^{1/2}}\right] \tag{28}$$

which are also the impulse amplitudes obtained by the solution of the three-impulse *shaped-input* technique.

From (20), we have

$$\omega = \frac{1}{T}\cos^{-1}\left(-\frac{A_1}{2}\exp(\sigma T)\right) \tag{29}$$

Substituting (29) into (16), we have

$$A_0 + A_1\exp(-\sigma T)\left(-\frac{A_1}{2}\exp(\sigma T) + \exp(-2\sigma T)\left(2\left(-\frac{A_1}{2}\exp(\sigma T)\right)^2 - 1\right) = 0 \tag{30}$$

which simplifies to

$$A_0 - \exp(-2\sigma T) = 0 \tag{31}$$

or

$$\sigma = -\frac{1}{2T}\ln(A_0) \tag{32}$$

Substituting T and A_0 from (26) and (27) respectively, we have

$$\sigma = -\xi_i\omega_i \tag{33}$$

and from (29), we have

$$\omega = \omega_i(1 - \xi_i^2)^{1/2} \tag{34}$$

which indicates that the use of either (19) or (20) leads to the same solution. Thus, the effect of the second delay is to provide a second infinite set of zeros, with all the zeros being coincident with the first infinite set, as indicated in Fig. 2. The controller should be normalized in the same fashion as the single time-delayed controller. This controller can also be represented as

$$\begin{aligned} u(s) &= (s^2 + 2\xi_i\omega_i s + \omega_i^2)^2(s^2 + 2\xi_i\omega_i s + 9\omega_i^2 - 8\xi_i^2\omega_i^2)^2 \\ &\quad \ldots (s^2 + 2\xi_i\omega_i s + n^2\omega_i^2 - (n^2 - 1)\xi_i^2\omega_i^2)^2 R(s) \\ &\qquad n = 1, 3, 5, \ldots \end{aligned} \tag{35}$$

Since the zeros are repeated, the controller transfer function should also be equal to

$$\frac{u(s)}{R(s)} = \left(\exp\left(\frac{\pi\xi_i}{(1 - \xi_i^2)^{1/2}}\right) + \exp(-sT)\right)^2 \tag{36}$$

or

$$\frac{u(s)}{R(s)} = \left(\exp\left(2\frac{\pi\xi_i}{(1 - \xi_i^2)^{1/2}}\right) + 2\exp\left(\frac{\pi\xi_i}{(1 - \xi_i^2)^{1/2}}\right)\exp(-sT) + \exp(-2sT)\right) \tag{37}$$

which is the same solution represented by (15), (27) and (28).

3. Proportional plus user selected multiple delay control

In the time-delay controller designed in § 2, the time-delay appears non-linearly in the equations, which increases the difficulty of arriving at closed-form solutions. The time-delay of the multiple time-delay controller is a function of the natural frequencies of the system. To eliminate this constraint, we propose, in this section, a class of time-delay controllers where the time-delay is selected by the user, thus giving greater latitude to the designer. The unknown gains of the time-delayed signals, which appear linearly, can be easily solved for. Consider an underdamped second-order plant and a two-time-delay controller (Fig. 4) where T is selected by the user. We choose the second time-delay to be twice that of the first although this is not necessary. We need to determine A_0, A_1 and A_2 to cancel the poles of the system. We require that the poles of the system be one set of zeros of the controller. The equations

$$A_0 + A_1 \exp(-\sigma T)\cos(\omega T) + A_2 \exp(-2\sigma T)\cos(2\omega T) = 0 \tag{38}$$

$$A_1 \exp(-\sigma T)\sin(\omega T) + A_2 \exp(-2\sigma T)\sin(2\omega T) = 0 \tag{39}$$

where $\sigma = \xi_i \omega_i$ and $\omega = \omega_i(1 - \xi_i^2)^{1/2}$ are used to solve for A_0, A_1 and A_2. The requirement that the final value of the prefiltered signal be the same at the reference signal leads to the constraint

$$A_0 + A_1 + A_2 = 1 \tag{40}$$

Solving (38), (39) and (40), we have

$$A_0 = \frac{\exp(-2\sigma T)}{\exp(-2\sigma T) - 2\exp(-\sigma T)\cos(\omega T) + 1} \tag{41}$$

$$A_1 = \frac{-2\exp(-\sigma T)\cos(\omega T)}{\exp(-2\sigma T) - 2\exp(-\sigma T)\cos(\omega T) + 1} \tag{42}$$

$$A_2 = \frac{1}{\exp(-2\sigma T) - 2\exp(-\sigma T)\cos(\omega T) + 1} \tag{43}$$

The Bode diagram of this open loop controller can be seen to produce a notch at the system's natural frequency. Thus, any error in estimated frequency will lead to an oscillatory response to a step input.

To build in robustness into the time-delay controller, we require that, in addition to the cancellation of the poles of the system, the variation of (38) and (39) with respect to frequency be zero. This leads to a total of five equations with three unknowns. We therefore introduce two more delays in the controller whose transfer function is now

$$A_0 + A_1 \exp(-sT) + A_2 \exp(-2sT) + A_3 \exp(-3sT) + A_4 \exp(-4sT) \tag{44}$$

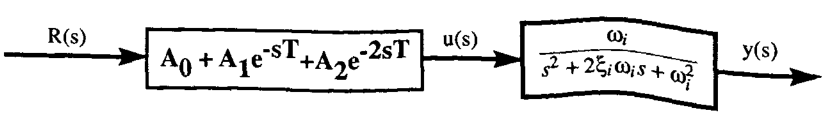

Figure 4. User selected time-delay controlled system.

The gains of the time delay controller can be determined such that the five equations (45)–(49)

$$A_0 + A_1 e^{-\sigma T}\cos(\omega T) + A_2 e^{-2\sigma T}\cos(2\omega T) + A_3 e^{-3\sigma T}\cos(3\omega T) + A_4 e^{-4\sigma T}\cos(4\omega T) = 0 \tag{45}$$

$$A_1 e^{-\sigma T}\sin(\omega T) + A_2 e^{-2\sigma T}\sin(2\omega T) + A_3 e^{-3\sigma T}\sin(3\omega T) + A_4 e^{-4\sigma T}\sin(4\omega T) = 0 \tag{46}$$

$$\frac{d}{d\omega}(A_0 + A_1 e^{-\sigma T}\cos(\omega T) + A_2 e^{-2\sigma T}\cos(2\omega T) + A_3 e^{-3\sigma T}\cos(3\omega T) + A_4 e^{-4\sigma T}\cos(4\omega T)) = 0 \tag{47}$$

$$\frac{d}{d\omega}(A_1 e^{-\sigma T}\sin(\omega T) + A_2 e^{-2\sigma T}\sin(2\omega T) + A_3 e^{-3\sigma T}\sin(3\omega T) + A_4 e^{-4\sigma T}\sin(4\omega T)) = 0 \tag{48}$$

$$A_0 + A_1 + A_2 + A_3 + A_4 = 1 \tag{49}$$

are satisfied. It can be shown that two, two-time-delay controllers in series are equivalent to the resulting controller. The transfer function of the controller is

$$\frac{u(s)}{R(s)} = \left(\frac{\exp(-2\sigma T) - 2\exp(-\sigma T)\cos(\omega T)\exp(-sT) + \exp(-2sT)}{\exp(-2\sigma T) - 2\exp(-\sigma T)\cos(\omega T) + 1}\right)^2 \tag{50}$$

It is to be noted that we will arrive at the same solution if we force the variations of (38) and (39) with respect to σ, to zero. Thus, the robustness is with respect to errors in estimated frequency and damping.

3.1. *Signs of the time-delay gains*

It might sometimes be required to limit the magnitude of the time-delay gains to be positive, while satisfying (40). This is desirable from a practical standpoint as we would not require the actuator to track large steps, which could occur if the time-delay gains were unconstrained. Hence, we require from (42), that the time-delay T satisfy

$$\frac{\pi}{2} < \omega T < \frac{3\pi}{2} \tag{51}$$

which is equivalent to

$$\frac{T_s}{4} < T < \frac{3T_s}{4} \tag{52}$$

where T_s is the system period. To illustrate that (41), (42) and (43) represent a generalization of the time-delay control (Singh and Vadali 1993), we study the case when $T = T_s/4$. This leads to

$$A_0 = \frac{\exp\left[\dfrac{\xi\pi}{(1-\xi^2)^{1/2}}\right]}{\exp\left[\dfrac{\xi\pi}{(1-\xi^2)^{1/2}}\right] + 1} \tag{53}$$

$$A_1 = 0 \tag{54}$$

$$A_2 = \frac{1}{\exp\left[\dfrac{\xi\pi}{(1-\xi^2)^{1/2}}\right] + 1} \tag{55}$$

and the controller degenerates to the two-impulse shaped input controller. Further, when $T = T_s/2$, we have

$$A_0 = \frac{\exp\left(\dfrac{2\xi\pi}{(1-\xi^2)^{1/2}}\right)}{\exp\left(\dfrac{2\xi\pi}{(1-\xi^2)^{1/2}}\right) + 2\exp\left(\dfrac{\xi\pi}{(1-\xi^2)^{1/2}}\right) + 1} \tag{56}$$

$$A_1 = \frac{2\exp\left(\dfrac{\xi\pi}{(1-\xi^2)^{1/2}}\right)}{\exp\left(\dfrac{2\xi\pi}{(1-\xi^2)^{1/2}}\right) + 2\exp\left(\dfrac{\xi\pi}{(1-\xi^2)^{1/2}}\right) + 1} \tag{57}$$

$$A_2 = \frac{1}{\exp\left(\dfrac{2\xi\pi}{(1-\xi^2)^{1/2}}\right) + 2\exp\left(\dfrac{\xi\pi}{(1-\xi^2)^{1/2}}\right) + 1} \tag{58}$$

which is equivalent to the three impulse shaped-input controller. It can also be shown that when $T = 3T_s/4$, the resulting controller is the same as when $T = T_s/4$ except that the system response delay is increased by one period.

3.2. *Periodicity*

The time-delay controller leads to an infinite number of zeros, one pair of which cancels a pair of poles of the system. To determine the zeros of the time-delay controller as a function of the user selected time T, we rewrite (38) and (39) with the appropriate values of A_0, A_1 and A_2, to arrive at

$$\exp(-2\sigma_s T) - 2\exp(-\sigma_s T)\cos(\omega_s(1-\xi_s^2)^{1/2}T)\exp(-\sigma T)\cos(\omega T) + \exp(-2\sigma T)\cos(2\omega T) = 0 \tag{59}$$

and

$$-2\exp(-\sigma_s T)\cos(\omega_s(1-\xi_s^2)^{1/2}T)\exp(-\sigma T)\sin(\omega T) + \exp(-2\sigma T)\sin(2\omega T) = 0 \tag{60}$$

From (59), which is quadratic in $\exp(-\sigma T)$, we have

$$\exp(-\sigma T) = \exp(-\sigma_s T) \times \frac{\cos(\omega_s(1-\xi_s^2)^{1/2}T)\cos(\omega T) \pm [\cos^2(\omega_s(1-\xi_s^2)^{1/2}T)\cos^2(\omega T) - \cos(2\omega T)]^{1/2}}{\cos(2\omega T)} \tag{61}$$

and from (60), we have

$$-2\cos(\omega_s(1-\xi_s^2)^{1/2}T)\sin(\omega T)+\frac{\cos(\omega_s(1-\xi_s^2)^{1/2}T)\cos(\omega T)\pm[\cos^2(\omega_s(1-\xi_s^2)^{1/2}T)\cos^2(\omega T)-\cos(2\omega T)]^{1/2}}{\cos(2\omega T)}\times\sin(2\omega T)=0 \quad (62)$$

Solving (61) and (62), we have

$$\sigma=\sigma_s \quad (63)$$

and

$$\omega=\pm\omega_s(1-\xi_s^2)^{1/2}+\frac{2n\pi}{T} \quad (64)$$

Thus, the smaller the value of T, the larger are the intervals between zeros of the time-delay controller. In the special case when $T=\pi/\omega_s(1-\xi_s^2)^{1/2}$ the imaginary parts of the zeros of the controller are odd multiples of the imaginary parts of the system poles.

The time-delay filter can be mapped into the z-domain, where one can show that the same objective is met, i.e. cancellation of the poles of the system by the time-delay filter zeros. However, in the z-domain, we would solve a polynomial equation to arrive at the zeros of the filter, which are finite in number, unlike in the s-domain where we solve a transcendental equation leading to an infinite set of zeros of the filter.

4. Time-delay control of multi-mode systems

4.1. *User selected time-delay*

The design of the time-delay controller in § 3 was simplified as the unknowns A_i appeared linearly in the constraint equations. This concept can easily be extended to the design of systems with more than one mode of vibration. The easiest design would be to connect the time-delay controller for each of the modes in series. This, however, will not lead to the minimum number of delays in the controller. In addition, the system response delay will not be the smallest possible. We can reformulate the constraints so that the selection of the gains of the time-delay controller satisfy the constraints simultaneously. To control one mode, we required a two-time-delay controller, and we need to add two more time-delays for the control of each additional mode. The transfer function of the time-delay controller for an m-mode system is

$$\sum_{i=0}^{2m}A_i\,e^{-isT} \quad (65)$$

The constants A_i have to satisfy the constraints

$$\sum_{i=0}^{2m}A_i\,e^{-i\sigma_jT}\cos(i\omega_jT)=0 \quad (66)$$

$$\sum_{i=0}^{2m}A_i\,e^{-i\sigma_jT}\sin(i\omega_jT)=0 \quad (67)$$

for $j=1$ to m, where σ_j is damping constant and ω_j is the damped natural frequency of the jth mode. We also require

$$\sum_{i=0}^{2m}A_i=1 \quad (68)$$

We thus have $2m+1$ equations in $2m+1$ unknowns, which can be solved easily as the unknowns appear linearly in the equations. The matrix equation to be solved is

$$\begin{bmatrix} 1 & e^{-\sigma_1 T}\cos(\omega_1 T) & e^{-2\sigma_1 T}\cos(2\omega_1 T) & \dots & e^{-2m\sigma_1 T}\cos(2m\omega_1 T) \\ 0 & e^{-\sigma_1 T}\sin(\omega_1 T) & e^{-2\sigma_1 T}\sin(2\omega_1 T) & \dots & e^{-2m\sigma_1 T}\sin(2m\omega_1 T) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & e^{-\sigma_m T}\cos(\omega_m T) & e^{-2\sigma_m T}\cos(2\omega_m T) & \dots & e^{-2m\sigma_m T}\cos(2m\omega_m T) \\ 0 & e^{-\sigma_m T}\sin(\omega_m T) & e^{-2\sigma_m T}\sin(2\omega_m T) & \dots & e^{-2m\sigma_m T}\sin(2m\omega_m T) \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

$$\times \begin{Bmatrix} A_0 \\ A_1 \\ \dots \\ A_{2m-2} \\ A_{2m-1} \\ A_{2m} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{69}$$

For certain values of T, the above matrix might become singular. In such cases, the solution can be obtained by using the pseudo-inverse technique. The pseudo-inverse solution will be exact for a row rank deficiency, but for a column rank deficiency, a least square approximation solution results, which does not satisfy the constraints. In this case, we need to select a different T so that column rank deficiency is avoided.

4.2. *Minimum time-delay*

In § 4.1, a set of linear equations had to be solved to arrive at the time-delay controller. We choose to modify the proposed technique such that all the time-delay gains are positive. When the gains are unconstrained, the resulting control input could 'ring' the unmodelled dynamics of the system. This is undesirable for flexible structures. With all the time-delay gains being positive, we are assured that they will lie in the range 0–1. We reformulate the design process so as to minimize the time-delay T, i.e.

$$\min J = T^2 \tag{70}$$

subject to the equality constraints

$$\textstyle\sum_{i=0}^{2m} A_i e^{-i\sigma_j T}\cos(i\omega_j T) = 0, \qquad j = 1 \text{ to } m \tag{71}$$

$$\textstyle\sum_{i=0}^{2m} A_i e^{-i\sigma_j T}\sin(i\omega_j T) = 0, \qquad j = 1 \text{ to } m \tag{72}$$

$$\textstyle\sum_{i=0}^{2m} A_i = 1 \tag{73}$$

and the inequality constraints

$$A_i \geqslant 0 \tag{74}$$

Solution of the optimization problem leads to the desired time-delay controller. The use of multiple time-delay controllers in cascade adds robustness to errors in system parameters.

5. Numerical examples

Two numerical examples are presented to illustrate the time-delay controller proposed in this paper. The first example is a spring–mass system (Fig. 5). Selecting the mass of the system to be 1 kg and the stiffness of the spring to be $1\,\mathrm{N\,m^{-1}}$, we have the following transfer function for the system

$$y(t) = \frac{1}{s^2 + 1}u(t), \quad T_s = 2\pi \tag{75}$$

We require the mass to be displaced from its initial position to a final position of unity, i.e. the input to the system is a unit step. We study the response of the system when we prefilter the step input to the system through two different time-delay filters, the first where the smallest time delay T, is one quarter the period of the natural frequency of the system T_s, the second where the time-delay T is $1/6T_s$. For each case, we also simulate the robust controller.

Figure 6(a), illustrates the prefiltered reference input to the system when the time-delay is chosen to be $T_s/4$. This leads to a controller that is exactly the same as the two-impulse shaped-input controller. The dashed curve corresponds to the non-robust controller and the solid line to the robust controller. The deflection of the mass shows very little residual vibration. In all the figures that follow, the dashed curve and the solid curve represent the non-robust and robust controllers, respectively.

We next choose $T = T_s/6$ seconds, which should force the system to its final position quicker than the previous controller. Figure 6(b), illustrates the prefiltered reference input to the system. It can be seen that the gains of the time-delayed signals are not all positive, thus forcing the reference input to go negative in the robust case. The response of the mass shows that the non-robust controller forces the mass to the desired position directly, unlike the robust case where the mass moves in a direction leading to an increase in error for a short interval of time before reaching the desired position.

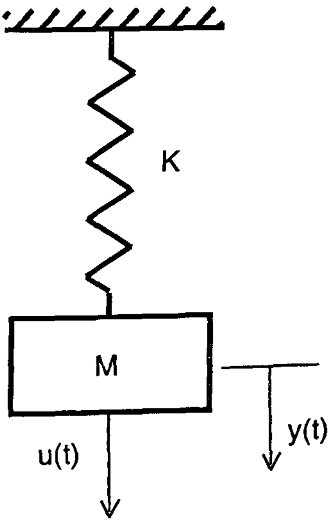

Figure 5. Spring mass system.

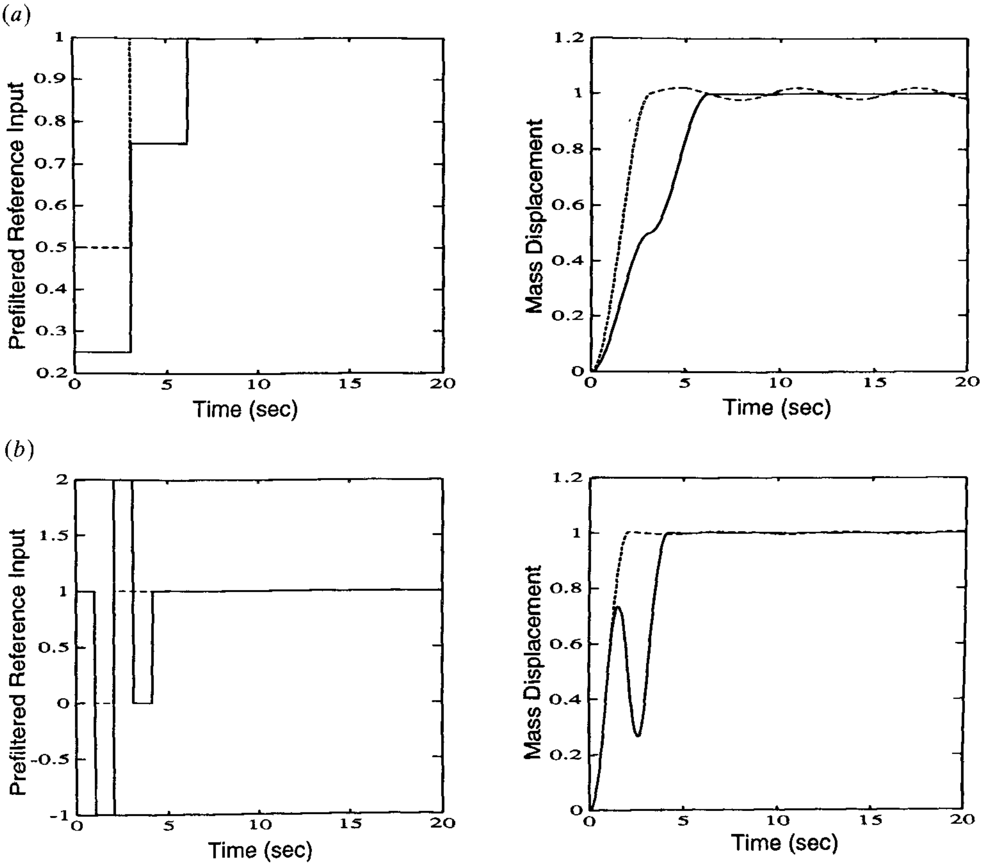

Figure 6. Prefiltered reference input and corresponding system response: (a) time-delay = $T_s/4$; (b) time-delay = $T_s/6$.

To compare the insensitivity of the two controllers to errors in estimated parameters, we plot their Bode diagrams. Figure 7(a), is the Bode diagram for the controller with positive time-delay gains. It can be seen that the magnitude plot of the non-robust controller is an envelope to the robust case. The robust controller has a slope of zero at odd multiples of the natural frequency of the system, unlike the non-robust case, which has a notch at these frequencies. The user-selected time-delay controller which leads to some of the time-delay gains being negative, shows a distinctly different behaviour (Fig. 7(b)). The Bode diagram of the robust controller, despite having a slope of zero at the system frequency, has a large magnitude at a frequency of 3. Thus, if the error in the estimated parameter was small, the robust controller would do a better job than the non-robust case, but after a critical value corresponding to the intersection of the two curves, the 'robust' controller will lead to a larger residual vibration than the 'non-robust' controller. It should also be noted that the next pole cancelled corresponds to a frequency of 5. This is expected as (64) shows that the periodicity of the controller increases with decreasing time-delay.

The next example is a single-link flexible-arm robot (Fig. 8) considered to demonstrate the use of the time-delay controller to attenuate two modes. The

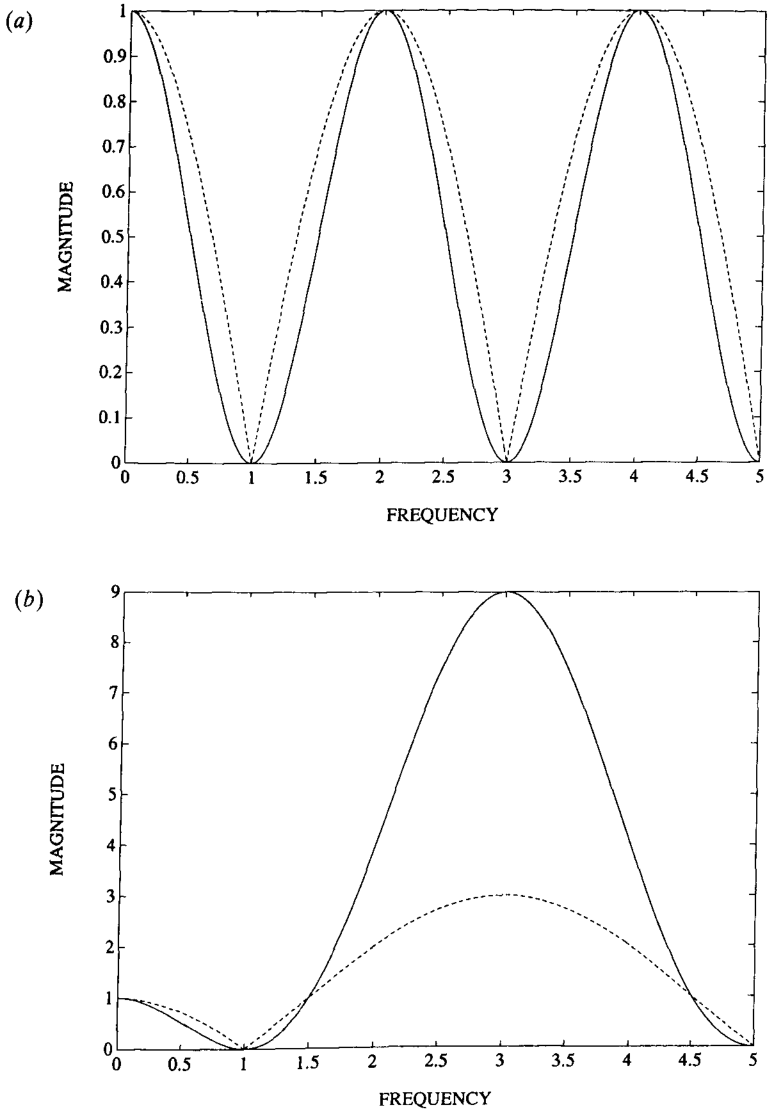

Figure 7. Bode diagram of time-delay controller for a single-mode system: (*a*) time-delay = $T_s/4$; (*b*) time-delay = $T_s/6$.

equations of motion of the robot considering only the first mode of vibration are

$$\begin{bmatrix} 0{\cdot}1128 & 0{\cdot}262095 \\ 0{\cdot}262095 & 0{\cdot}63466 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\zeta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 612{\cdot}821 \end{bmatrix} \begin{Bmatrix} \theta \\ \zeta \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} u \tag{76}$$

The length of the robot arm is 0·726 m. We use a simple proportional feedback controller

$$u = -[100 \quad 0 \quad 0 \quad 0] \begin{Bmatrix} \theta - \theta_{\text{ref}} \\ \zeta \\ \dot{\theta} \\ \dot{\zeta} \end{Bmatrix} \tag{77}$$

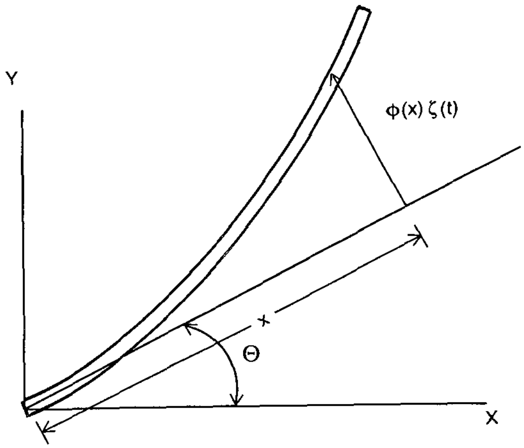

Figure 8. Single flexible-link robot.

so that the eigenvalues of the closed loop system are

$$s = \pm j21{\cdot}6 \quad \text{and} \quad \pm j212{\cdot}59 \tag{78}$$

We design the time-delay controller to prefilter the reference input to the system so as to produce a non-oscillatory response. The reference input to the system is a unit step of one radian. The first controller is the user selected time-delay controller, with a time-delay of 0·05 s. The selected time delay leads to time-delay gains that are both positive and negative (Table 1).

Figure 9(*a*), is the prefiltered reference input to the flexible robot. The solid curve and the dashed curve correspond to the robust and non-robust controllers respectively. The change in signs of the steps is the effect of the negative gains of the time-delayed signals. At the time of the final time-delay (0·2 s and 0·4 s for the non-robust and robust case respectively) it can be seen that the tip has moved to its desired position of 0·726 m. For the next design, the time-delay is optimized with the constraint that all the time-delay gains have to be positive. The optimization toolbox of Matlab (Grace 1990) which uses the Sequential Quadratic Programming (SQP) method is used to arrive at the optimal time and time-delay gains, which are listed in Table 2. The final delay in the controller is incidentally smaller than that of the previous controller. We are however at liberty to choose a time-delay smaller than 0·0402, if we are not constrained by the requirement that the time-delay gains have to be all positive. Figure 9(*b*) shows that the prefiltered input reference steps do not have any sign changes

	Time-delay gains	Time-delay
A_0	0·3479	0
A_1	−0·0786	0·05
A_2	0·4614	2*0·05
A_3	−0·0786	3*0·05
A_4	0·3479	4*0·05

Table 1. User selected time-delay.

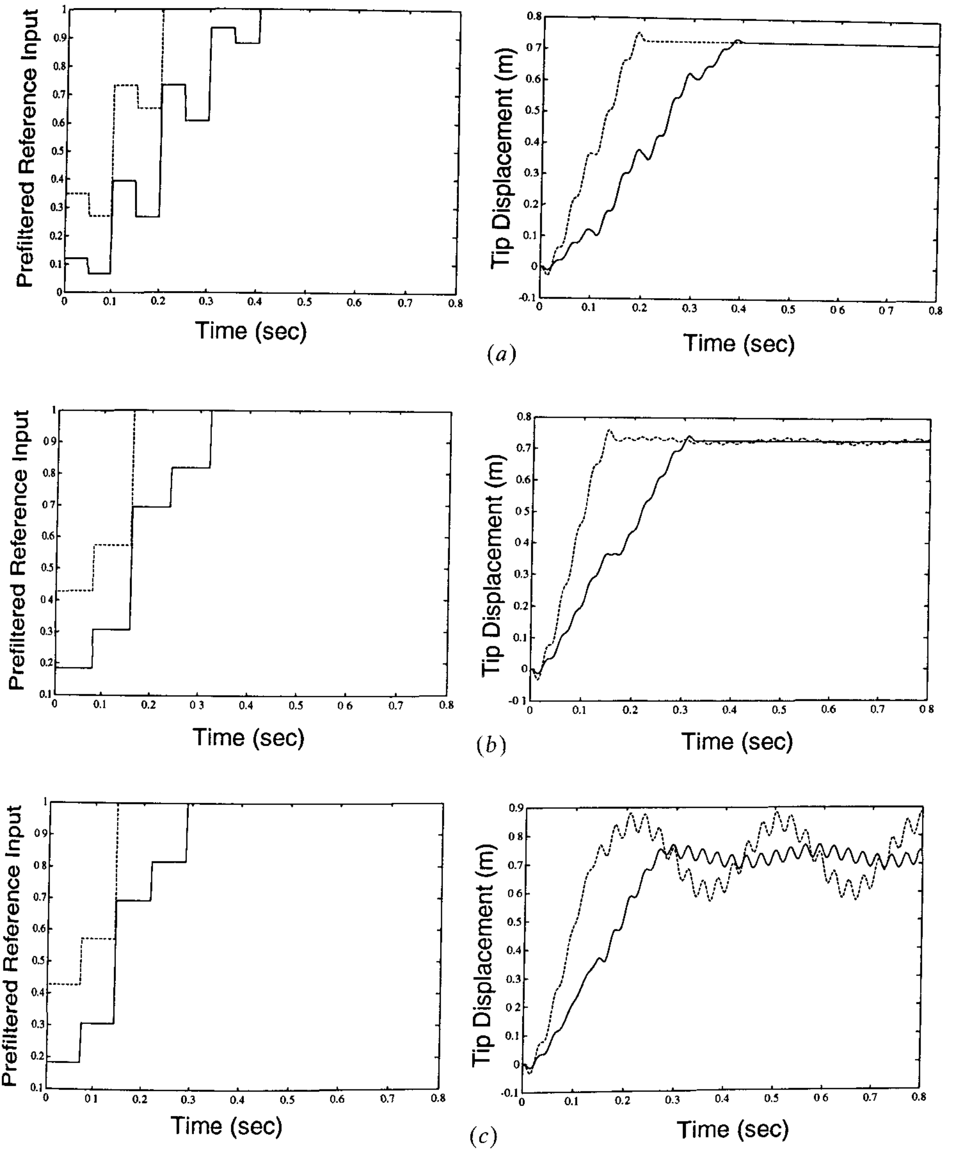

Figure 9. Prefiltered reference input and corresponding system response of flexible-link robot: (*a*) user selected time-delay; (*b*) minimum time-delay; (*c*) minimum time-delay with +10% error in frequency.

	Time-delay gains	Time-delay
A_0	0·42825	0
A_1	0	0·0402
A_2	0·14351	2*0·0402
A_3	0	3*0·0402
A_4	0·42825	4*0·0402

Table 2. Minimum time-delay.

and that the response of the system has a characteristic that is similar to the previous case with a smaller high frequency amplitude superimposed on the gross motion. Figure 10 illustrates the feasible region for the gains of the optimized time-delay controller. It can be seen for the present example that the proportional gain A_0, and the gain of the largest time-delay signal A_4 are the same, as are the gains of the smallest time-delay signal A_1 and A_3. The optimized time-delay controller corresponds to the situation when the gains A_1 and A_3 are zero. We next design a controller with +10% error in both the frequencies. The prefiltered reference input is the same as the previous controller except for the time-delay of 0·0366 s, as shown in Table 3. Figure 9(c), illustrates that both the non-robust and robust controller have considerable residual vibration, although only the high frequency manifests itself in the robust case as compared with the non-robust controlled system where both the frequencies are conspicuous in their presence.

Bode diagrams (magnitude) are used to highlight the differences in time-delay controllers when both positive and negative gains are used and when only

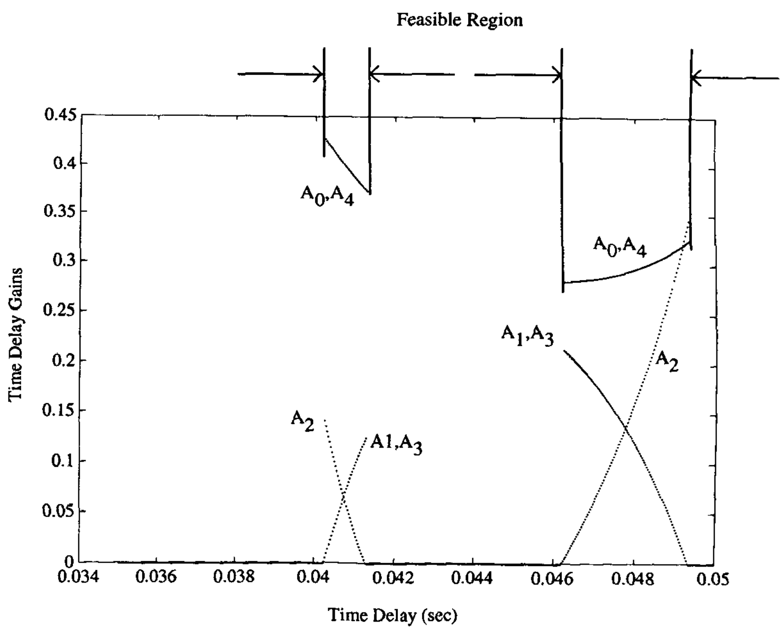

Figure 10. Feasible regions for the positive-gain/minimum time-delay controller.

	Time-delay gains	Time-delay
A_0	0·42825	0
A_1	0	0·0366
A_2	0·14351	2*0·0366
A_3	0	3*0·0366
A_4	0·42825	4*0·0366

Table 3. Minimum time-delay with +10% error in frequencies.

positive gains are used. Figure 11(*b*) helps confirm the fact that the Bode diagram of the robust time-delay controller with positive gains is enclosed within that of the non-robust case. The magnitude curve goes to zero at the system natural frequencies of 21·6 and 212·59 rad s^{-1}. When the time-delay gains are both positive and negative, it can be seen that large errors in system parameters can lead to large residual vibration amplitudes (Fig. 11(*a*)).

6. Conclusions

Attenuation of residual vibration of flexible structures, in an open loop fashion, by cancelling the underdamped poles of the system via a time-delay

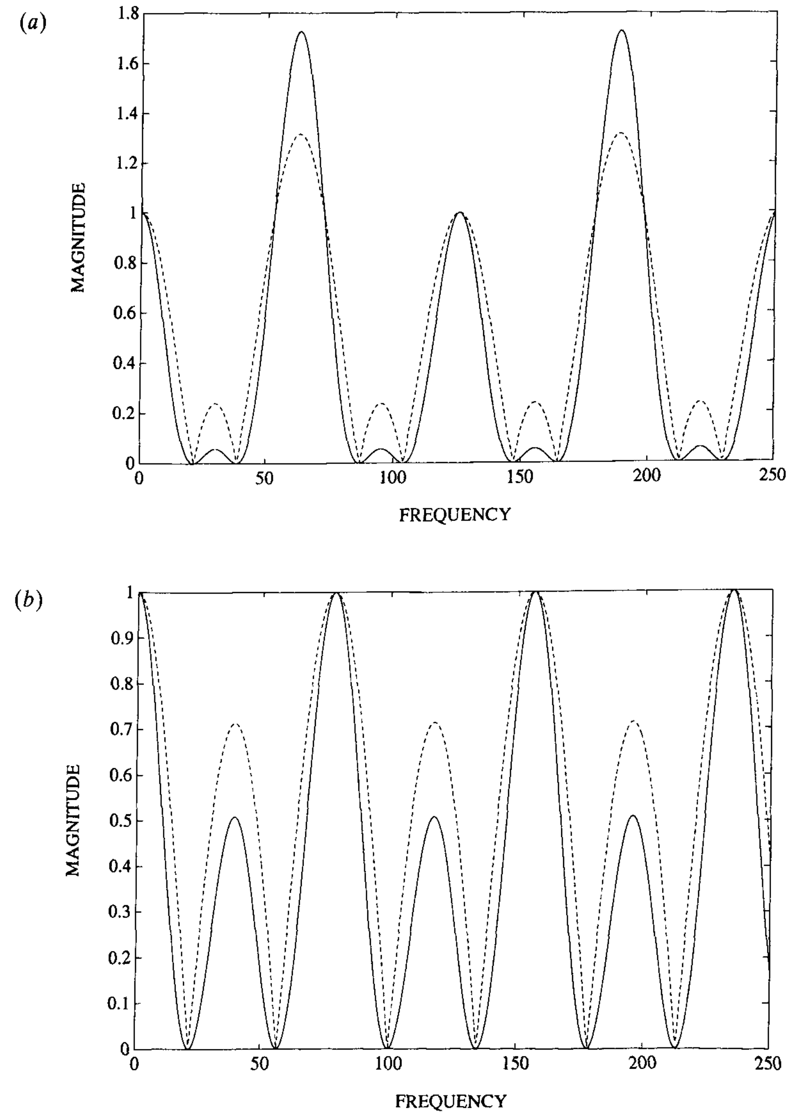

Figure 11. Bode diagram of time-delay controller for a multiple mode system: (*a*) user selected time-delay; (*b*) minimum time-delay.

controller has led to results that are the same as the shaped input control technique for a single-mode system. The design of the time-delay controller is generalized so that the time-delay can be selected by the user and is not dictated by the system dynamics. This controller has a simple structure, and the solution can be obtained by solving a linear algebraic equation. This procedure has been extended to design controllers for systems with multiple modes. The proposed final controller involves minimizing the time-delay, with the time-delay gains constrained to lie in the range 0 and 1.

Two simple examples, a spring–mass system and a single-link flexible-arm robot are used to demonstrate the effectiveness of the proposed technique. It has been shown that when gains of the time-delay controllers are constrained to be positive, the manoeuvre is completed in an optimal time, which is a function of the system's natural frequencies, unlike the case where the time-delays are selected by the user. The controller with positive gains has the advantage of relatively smaller residual vibration, even when the estimated parameters of the system have large errors.

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