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# Exact Time-Optimal Control of the Wave Equation

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**The time-optimal control of a distributed parameter system is derived in closed form. The class of systems studied in this work is distributed parameter systems whose dynamics are governed by the wave equation. A frequency domain approach is utilized to arrive at the time-optimal solution that is bang-off-bang. To corroborate the optimality of the control profile derived for the distributed parameter system, the system is discretized in space and a series of time-optimal control problems is solved for the finite dimensional model, with an increasing number of flexible modes. The limiting controller shows the convergence of the first and last switch of the bang-bang controller of the finite dimensional system to the first and last switch of the bang-off-bang controller of the distributed parameter system, in addition to the convergence of the maneuver time. The number of switches in between the first and last switch is a function of the order of the finite dimensional system. The maneuver time of the distributed parameter system is compared with that of an equivalent rigid system, and it is shown for certain maneuvers that the bang-bang control profile of the rigid system is also the time-optimal control of the distributed system.**

## I. Introduction

IN recent years, interest in the study of large space structures has grown and is reflected by the publication of numerous papers dealing with the modeling and control aspects of these structures. The effects of flexibility of these structures have to be taken into account in designing controllers. Most of the papers arrive at tractable models by spatial discretization of the structure using finite element or the assumed mode method. Designs of time-optimal controllers for these reduced-order models have been shown to be bang-bang. Singh et al.[1] solve for the time-optimal control profile of a flexible slewing beam by deriving a set of algebraic equations that are satisfied by the switch times and the final time. A homotopy approach is used to solve for the switch times and the maneuver time. Ben-Asher et al.[2] arrive at the time-optimal control of a slewing beam using a parameter optimization technique. The model they use to represent the dynamics of the system is arrived at by discretizing the system by the assumed mode method. They also show that the control profile is antisymmetric about the midmaneuver time for a system that is undamped. Singh and Vadali[3] present a frequency domain approach to arrive at the time-optimal control profile for a system represented by a system of ordinary differential equations. A parameter optimization problem is formulated to minimize the maneuver time subject to the constraint that the time-delay filter used to generate the bang-bang control profile cancels all of the poles of the system.

Bennighof and Boucher[4] solve the minimum-effort problem for a system with a finite number of modes with the objective of minimizing the excitation of the uncontrolled higher modes. They show that the spillover energy cannot be decreased for time less than a constant that corresponds to the time required for waves to travel through the structure. They infer from this result that the minimum time control of a structure is a function of the time required for the wave to travel between actuators.

Bennighof and Boucher[5] confirm their conjecture that the wave speed defines the minimum time required for a maneuver to be completed. They solve the minimum time control problem for a rest-to-rest maneuver of a one-dimensional second-order distributed parameter system that is driven by two control inputs. They use the traveling wave formulation to solve the time-optimal controller exactly and show that it is bang-off-bang and not bang-bang, which is the time-optimal solution of a normal lumped parameter system.

This paper addresses the problem of the design of a time-optimal controller for systems represented by the wave equation. Axial vibration of rods, transverse vibration of a string, and torsional vibration of a rod are all represented by the wave equation. The only control input available is at one end of the axially vibrating rod. A frequency domain approach is used to arrive at the minimum-time controller in closed form.

In this work a uniform rod without structural damping is considered. The Euler–Lagrange equation of motion of the rod in axial vibration is the wave equation. Laplace transformation of the wave equation leads to a transcendental transfer function relating the displacement of any point on the rod to an input applied at one end. The poles defining the dynamics of the system can be easily identified from the transfer function, which will be used in the design of the time-optimal controller.

To verify the results, the partial differential equation representing the dynamics of the rod is discretized leading to a finite order transfer function representing the dynamics of the system. The time-optimal control profile is derived for a series of discretized models, each with a large number of modes used to represent the system dynamics. The minimum-time control profile is bang-bang with $(2n + 1)$ switches for a system whose flexible body motion is represented by $n$ modes. Parameter optimization is used to obtain the switch times, and optimality is verified by the maximum principle. The maneuver time is then compared with that predicted by the closed-form solution to prove that the bang-off-bang control profile is indeed optimal. Finally, the maneuver time of the wave equation is compared with that of a rigid-body system for different maneuvers.

## II. Problem Formulation

A flexible rod undergoing rest-to-rest motion is used to illustrate the procedure to derive a time-optimal controller for distributed parameter systems in the frequency domain. For simplicity we assume that the rod has uniform properties in the axial direction, and control input is located at one end of the rod. The partial differential equation of motion of the rod is

$$\rho \frac{\partial^2 y(x,t)}{\partial t^2} = EA \frac{\partial^2 y(x,t)}{\partial x^2} \tag{1}$$

where $\rho$ is the mass per unit length, $E$ is the Young's modulus, and $A$ is the cross-sectional area. The boundary conditions of the system are

$$\frac{\partial y(L,t)}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial y(0,t)}{\partial x} = -\frac{u(t)}{EA} \tag{2}$$

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where $L$ is the length of the beam and the initial and final conditions are

$$y(x,0) = \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=t_f} = 0 \qquad y(x,t_f) = y_f, \qquad x \in [0, L] \tag{3}$$

which imply a rest-to-rest maneuver. Laplace transformation of Eq. (1) leads to

$$y''(x,s) - c^2 s^2 y(x,s) = 0 \tag{4}$$

where $(\cdot)'$ indicates the partial derivative of $(\cdot)$ with respect to the spatial coordinate and

$$c^2 = \rho/EA \tag{5}$$

where $1/c$ is the wave speed. The transformed boundary conditions are

$$y'(L,s) = 0 \qquad \text{and} \qquad y'(0,s) = -[u(s)/EA] \tag{6}$$

where the control input is constrained to lie in the range

$$-U \le u(t) \le U \tag{7}$$

The solution of the differential equation (4) is

$$y(x,s) = c_1 \cosh(csx) + c_2 \sinh(csx) \tag{8}$$

The constants $c_1$ and $c_2$ are obtained by evaluating the boundary conditions [Eq. (6)] at $x = 0$ and $L$. The transfer function of the system can now be represented as

$$\frac{y(x,s)}{u(s)} = \frac{\cosh[cs(L-x)]}{csEA\sinh(csL)} \tag{9}$$

The transcendental transfer function indicates that the system response is defined by an infinite number of modes. The time response of the system can be derived by the inverse Laplace transform of Eq. (9), which is

$$y(x,t) = \int_0^t \frac{1}{EAc}\left\{\frac{\tau^2}{2cL} + \frac{2cL}{\pi^2}\Sigma_{n=1}^{\infty}\frac{(-1)^n}{n^2} \times \cos\left[\frac{n\pi(L-x)}{L}\right]\left[1-\cos\left(\frac{n\pi\tau}{cL}\right)\right]\right\}u(t-\tau)\,\mathrm{d}\tau \tag{10}$$

## III. Time-Optimal Control Design

Singh and Vadali[3] have shown that the time-optimal control profile for a finite dimensional system can be reformulated as the design of a time-delay filter, a set of zeros of which cancel all of the poles of the system. The time-delay filter is constrained to produce a bang-bang output when it is subject to a step input. We adopt the same approach to derive the time-optimal control profile for the distributed parameter system. Following Bennighof and Boucher,[5] who state that the time-optimal control profile for distributed parameter systems is bang-off-bang rather than bang-bang, we state our optimization problem as the design of a time-delay filter that generates a bang-off-bang profile when it is subject to a step input as shown in Fig. 1. This bang-off-bang profile represents the parametrized open-loop time-optimal control.

The structure of the time-delay filter is selected to generate a control profile that is antisymmetric about the midmaneuver time. This follows from a result presented in a paper by Singh and Vadali,[3] where the authors illustrate that the time-optimal control profile for an undamped system is antisymmetric about the midmaneuver time, which reduces the number of parameters to be solved for. They further show that the antisymmetric profile cancels the rigid-body poles of the system, thus satisfying the velocity boundary conditions for a rest-to-rest maneuver. The transfer function of the time-delay filter is

$$G = 1 - e^{-s(T_2-T_1)} - e^{-s(T_2+T_1)} + e^{-2sT_2} \tag{11}$$

where $T_2$ is the midmaneuver time and $T_2 - T_1$ is the first switch time. The zeros of the time-delay filter have to cancel all of the poles of the transfer function of the distributed parameter system [Eq. (9)], which can be rewritten using product expansion as[6]

$$\frac{y(x,s)}{u(s)} = \frac{1}{EAcs}\frac{\Pi_{k=1}^{\infty}\{1+[4c^2s^2(L-x)^2]/[(2k-1)^2\pi^2]\}}{cLs\Pi_{k=1}^{\infty}[1+(c^2L^2s^2)/(k^2\pi^2)]} \tag{12}$$

Equation (12) indicates that the transfer function poles defined by

$$s_k = \pm(k\pi/cL)j \qquad k = 1, 2, \ldots, \infty \tag{13}$$

are located on the imaginary axis.

The zeros of the time-delay filter can be determined by equating the real and imaginary parts of Eq. (11) to zero, after substituting

$$s = j\omega \tag{14}$$

since we know that we are required to cancel poles that are located on the imaginary axis. The zeros are determined from the equations

$$1 - \cos[\omega(T_2 - T_1)] - \cos[\omega(T_2 + T_1)] + \cos(2\omega T_2) = 0 \tag{15}$$

and

$$-\sin[\omega(T_2 - T_1)] - \sin[\omega(T_2 + T_1)] + \sin(2\omega T_2) = 0 \tag{16}$$

which can be simplified to

$$2\cos(\omega T_2)[-\cos(\omega T_1) + \cos(\omega T_2)] = 0 \tag{17}$$

and

$$2\sin(\omega T_2)[-\cos(\omega T_1) + \cos(\omega T_2)] = 0 \tag{18}$$

respectively. Equations (15) and (16) can be derived in the time domain as well using the superposition principle.[2] Thus, we require

$$\cos(\omega T_2) = \cos(\omega T_1) \tag{19}$$

to satisfy Eqs. (17) and (18). The solution of Eq. (19) is

$$T_2 = \pm T_1 + (2kn\pi/\omega) \qquad k, n = 1, 2, 3, \ldots, \infty \tag{20}$$

We include two integer variables $k$ and $n$ into Eq. (20), since we intend to cancel the infinite modes

$$\omega = k\pi/cL \tag{21}$$

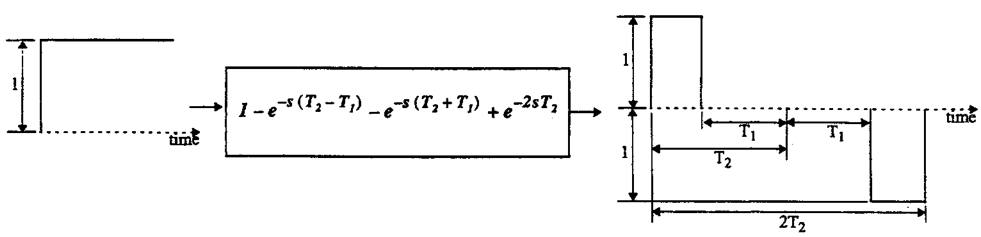


**Fig. 1 Time-delay filter.**

where $k$ ranges from 1 to $\infty$. The index $n$ is used to indicate the multiplicity of solutions of Eq. (19) that cancels all poles given by Eq. (21). Substituting Eq. (21) into Eq. (20), we have

$$T_2 = \pm T_1 + 2ncL \tag{22}$$

Assuming

$$T_2 = -T_1 + 2ncL \tag{23}$$

the bang-off-bang control profile can be represented as

$$u(s) = (U/s)\left[1 - e^{-2s(ncL - T_1)} - e^{-2sncL} + e^{-2s(2ncL - T_1)}\right] \tag{24}$$

We need to determine $T_1$ such that the boundary conditions of the rest-to-rest maneuver are met. Substituting Eq. (24) into Eq. (12) and applying the final value theorem, we have

$$\begin{aligned} y_f &= \lim_{t\to\infty} y(x,t) = \lim_{s\to 0} sy(x,s) \\ &= \lim_{s\to 0} = \frac{s}{EAcs} \frac{\Pi_{k=1}^{\infty}\{1 + [4c^2s^2(L-x)^2]/[(2k-1)^2\pi^2]\}}{cLs\Pi_{k=1}^{\infty}[1 + (c^2L^2s^2/k^2\pi^2)]} \\ &\times \frac{U}{s}\left[1 - e^{-2s(ncL - T_1)} - e^{-2sncL} + e^{-2s(2ncL - T_1)}\right] = \frac{0}{0} \end{aligned} \tag{25}$$

Applying L'Hospital rule twice, we can show that

$$T_1 = c[nL - (y_f EA/4Un)] \tag{26}$$

and

$$T_2 = c[nL + (y_f EA/4Un)] \tag{27}$$

where $y_f$ represents the final displacement.

Since $T_1$ is a positive number, this solution is valid only when

$$y_f < \frac{4LUn^2}{EA} \tag{28}$$

When Eq. (28) is violated, the second possible solution from Eq. (22) leads to

$$T_2 = T_1 + 2ncL \tag{29}$$

and the resulting control profile is

$$u(s) = (U/s)\left[1 - e^{-2sncL} - e^{-2s(ncL + T_1)} + e^{-2s(2ncL + T_1)}\right] \tag{30}$$

We can now solve for $T_1$ and $T_2$ using the procedure delineated earlier that results in

$$T_1 = c[-nL + (y_f EA/4Un)] \tag{31}$$

and

$$T_2 = c[nL + (y_f EA/4Un)] \tag{32}$$

Since the maneuver time is $T_f = 2T_2$, we see that, irrespective of whether $T_2 = -T_1 + 2ncL$ or $T_2 = T_1 + 2ncL$, the maneuver time is

$$T_f = c[2nL + (y_f EA/2Un)] \tag{33}$$

To illustrate the two solutions predicted by the frequency domain approach, we assume that $\rho$, $EA$, $L$, and $U$ are equal to 1. Results of the simulation of Eq. (10) for the case when $y_f = 1$ is shown in Fig. 2, which exemplifies the mechanism of motion of the flexible rod. A traveling wave is generated by the first pulse that travels along the length of the rod and is reflected at the uncontrolled end and is finally canceled by the second pulse of the time-optimal control profile. The parameters of the time-optimal control are defined by Eqs. (26) and (27).

For the case when $y_f = 5$, the parameters of the time-optimal control are defined by Eqs. (31) and (32). Results of the simulation of Eq. (10) indicate that the wave generated by the first pulse is reflected twice before it is canceled by the second pulse (Fig. 3).

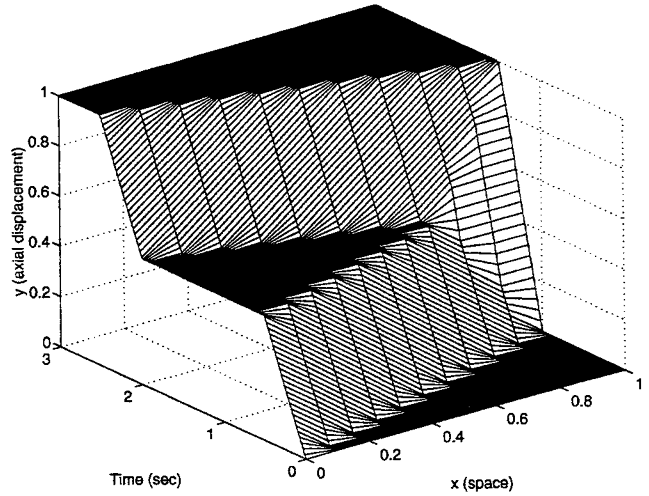


**Fig. 2 Time-optimal response for $y_f = 1$.**

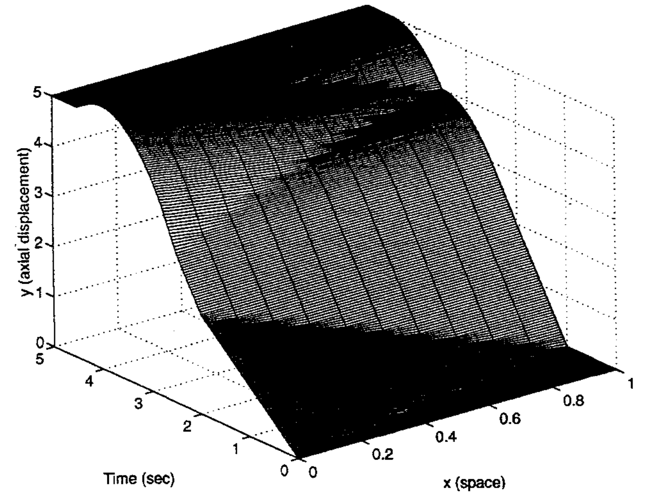


**Fig. 3 Time-optimal response for $y_f = 5$.**

## IV. Verification of Optimality

To corroborate the optimality of the closed-form time-optimal solution, we solve a series of discretized problems that in the limit should indicate the convergence of the first and last switch time and maneuver time to the exact solution. The assumed mode method is used to discretize the floating flexible rod whose governing vector differential equation of motion is

$$M\underline{\ddot{x}} + K\underline{x} = bu \qquad b, x \in \mathcal{R}^{n+1} \tag{34}$$

where $n$ is the number of flexible modes used to represent the system dynamics. The term $M$ is the mass, $K$ is the stiffness matrix, and $b$ is the control influence vector. To illustrate the optimality of the exact time-optimal controller, we assume that $\rho$, $EA$, $L$, and $U$ are equal to 1.

A parameter optimization problem is formulated for the discretized system as proposed by Singh and Vadali.[3] Since the discretized model of the flexible rod is normal,[7] the necessary conditions for optimality are also the sufficient conditions. It can be shown that the time-optimal control profile for an undamped system is antisymmetric about the midmaneuver time,[3] which leads to the reduction of the number of parameters to be optimized for. A series of optimal control problems for a rest-to-rest maneuver of $y_f = 1$ are solved with an increasing number of flexible modes, and the resulting control profiles are examined. Figure 4 illustrates the time-optimal control profiles for discretized systems where the number of flexible modes (Nfm) ranges from 1 to 8.

It is evident from Fig. 4 that the first switch time is converging to 0.5 s, the final switch time to 2.0 s, and the maneuver time to 2.5 s.

Table 1 Switch times and maneuver time of discretized systems

| $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ | $T_9$ | $T_f$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.7309 | 1.2003 | — | — | — | — | — | — | — | 2.4006 |
| 0.6367 | 0.9254 | 1.2352 | — | — | — | — | — | — | 2.4704 |
| 0.5902 | 0.7897 | 1.0139 | 1.2448 | — | — | — | — | — | 2.4896 |
| 0.5634 | 0.7097 | 0.882 | 1.0638 | 1.248 | — | — | — | — | 2.4960 |
| 0.5466 | 0.6583 | 0.7957 | 0.9436 | 1.0954 | 1.2492 | — | — | — | 2.4984 |
| 0.5355 | 0.6234 | 0.7357 | 0.8588 | 0.987 | 1.1178 | 1.2497 | — | — | 2.4994 |
| 0.5278 | 0.5987 | 0.6921 | 0.7966 | 0.9067 | 1.0198 | 1.1345 | 1.2499 | — | 2.4998 |
| 0.5223 | 0.5805 | 0.6595 | 0.7494 | 0.8451 | 0.9442 | 1.0453 | 1.1474 | 1.2499 | 2.4998 |

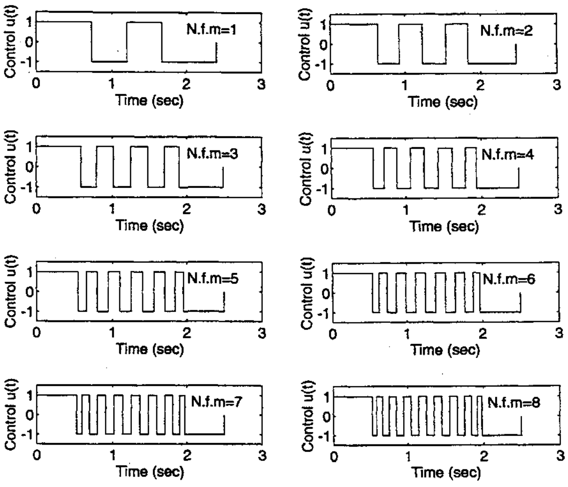


Fig. 4 Time-optimal control profiles for discretized system.

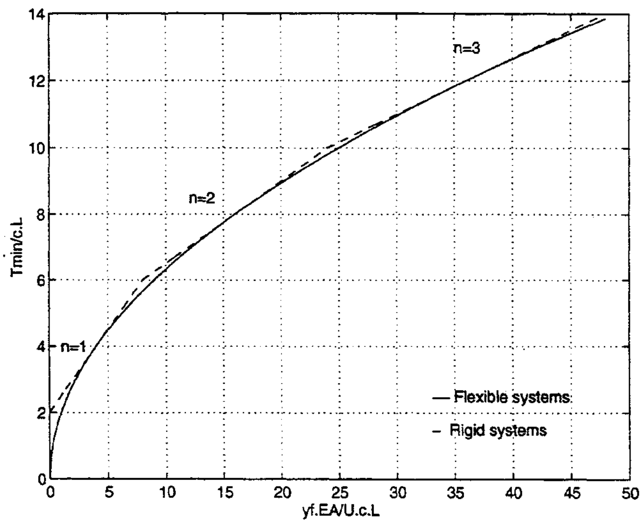


Fig. 5 Maneuver times for flexible and rigid systems.

In the interval from 0.5 to 2.0 s, an increasing number of switches exist with an increase in the number of modes used to represent the flexible body motion of the rod. In the limit, one can conjecture that there exists chattering at infinite frequency, since the Pontryagin maximum principle requires the time-optimal control profile of a normal system to be bang-bang. Thus, the average of the control in the interval between the first and last switch time is zero. Table 1 lists the optimal parameters of the time-delay filter that generates the time-optimal control for systems with one to eight flexible modes used to represent the system dynamics. It is evident from Table 1 that the first switch time is tending toward 0.5 and the maneuver time toward 2.5. Table 1 lists the switch times before the midmaneuver time and the final time.

For a final displacement $y_f$ equal to 1, the exact time-optimal control profile is characterized by the switch times and the maneuver time that are

$$T_1 = 0.5 \qquad T_2 = 2.0 \qquad \text{and} \qquad T_f = 2.5 \tag{35}$$

This confirms that the bang-off-bang control profile is the time-optimal control of the wave equation.

## V. Comparison with a Rigid Model

To compare the time-optimal controller of the wave equation with the time-optimal controller for a rigid system with the same mass, for a variety of maneuvers, we nondimensionalize the maneuver time of the wave equation. The maneuver time of the wave equation is

$$T_f = T_{\min} = c[2nL + (y_f EA/2Un)] \tag{36}$$

which can be rewritten as

$$\frac{T_{\min}}{cL} = \frac{y_f EA}{2UncL} + 2n \tag{37}$$

where the left-hand side of Eq. (37) is the nondimensionalized time and $n$ must satisfy the inequality

$$4n(n-1) \le (y_f EA/UL) \le 4n(n+1) \tag{38}$$

The time-optimal controller of a rigid system is bang-bang, and the maneuver time $T^R_{\min}$ can be shown to be

$$T^R_{\min} = 2\sqrt{(my_f/U)} \tag{39}$$

which can be rewritten as

$$\frac{T^R_{\min}}{cL} = 2\sqrt{\frac{my_f}{Uc^2L^2}} \tag{40}$$

where $m = \rho L$. Figure 5 illustrates the variation of the nondimensionalized maneuver time of the wave equation and the equivalent rigid system as a function of the nondimensionalized displacement. It can be seen that the curve representing the solution of the wave equation is tangential to the solution of the rigid case at discrete points, which indicates that for certain maneuvers the rigid-body solution is also the solution of the wave equation.

The parameters for which the solution of the wave equation is the same as that of the rigid case correspond to the situation when the first switch time is equal to the midmaneuver time. Therefore

$$T_1 = c[nL - (y_f EA/4Un)] = 0 \tag{41}$$

which leads to

$$y_f EA/UL = 4n^2 \tag{42}$$

Figure 5 indicates that the curve representing the nondimensionalized maneuver time is piecewise linear. The location of change of slope of the final time as a function of the maneuver can be determined by equating the solution of the maneuver time for successive values of $n$:

$$T_{\min} = c\left(2nL + \frac{y_f EA}{2Un}\right) = c\left[2(n+1)L + \frac{y_f EA}{2U(n+1)}\right] \tag{43}$$

which leads to

$$y_f EA/UL = 4n(n+1) \tag{44}$$

The effect of the application of the time-optimal controller of a rigid system with the same mass as the flexible system is studied

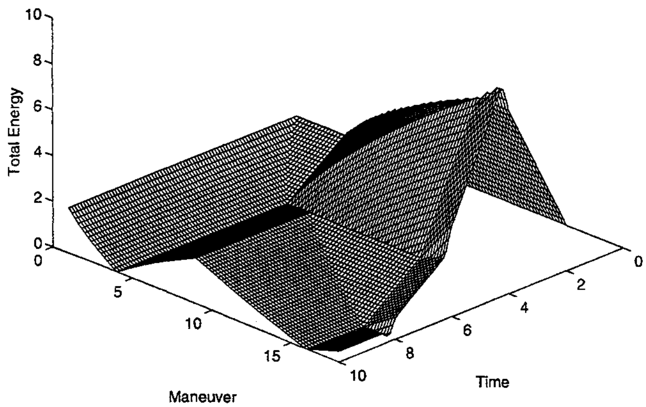


**Fig. 6 Variation of total energy of the flexible structure.**

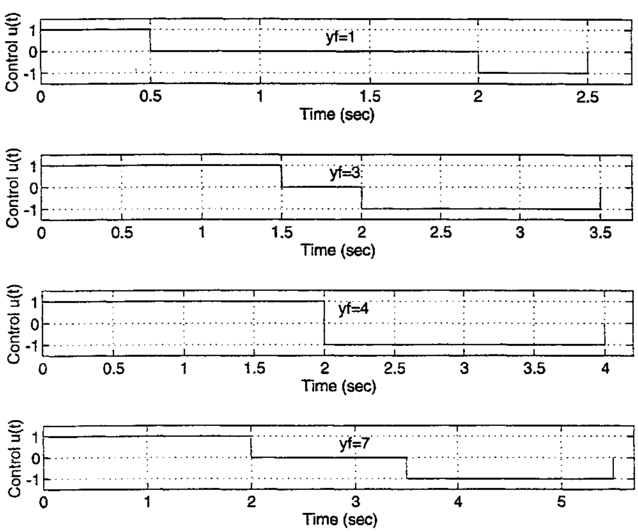


**Fig. 7 Variation of time-optimal control profile with maneuver.**

for a variety of maneuvers. Figure 6 illustrates the variation of the total energy of the system as a function of time for different maneuvers, when the flexible system is subject to a bang-bang controller designed to control the equivalent rigid system. It can be seen that the residual energy is a function of the maneuver. Only those maneuvers

$$y_f = \frac{4n^2UL}{EA} \qquad n = 1, 2, \ldots, \infty \tag{45}$$

where the time-delay filter, which generates a bang-bang output, also cancels all of the poles of the flexible system result in zero residual energy.

Figure 7 indicates the time-optimal control profile for different maneuvers. As the maneuver increases, the pulse width increases while the second switch time remains constant. For a maneuver that corresponds to the solution of Eq. (12), the first the second switches are coincident, leading to a bang-bang control profile as illustrated in Fig. 7 for $y_f = 4$. Any further increase in the maneuver leads to a control profile whose pulse width is constant, and the coasting period changes with the maneuver up to a maneuver indicated by Eq. (44) where a different value of $n$ defines the optimal controller.

## VI. Conclusions

A frequency domain approach has been used to solve for the time-optimal control of a distributed parameter system. This procedure requires the cancellation of the infinite poles of the distributed parameter system by the zeros of a time-delay filter whose output is the time-optimal control profile when it is subject to a step input. The time-optimal control profile for the rest-to-rest maneuver of a rod whose dynamics are governed by the wave equation is shown to be a bang-off-bang control profile that is antisymmetric about the midmaneuver time, which coincides with conclusions drawn by Bennighof and Boucher. The constraints requiring the zeros of the time-delay filter to cancel the poles of the system and satisfy the boundary conditions are used to solve for the parameters of the time-delay filter.

A series of discretized problems are solved that indicate that the bang-off-bang control profile is indeed the time-optimal control. A parametric study of the maneuver time vs the maneuver has indicated some interesting results that include the piecewise linear nature of the maneuver time as a function of the final displacement. Maneuvers for which the solution to the rigid model is the solution of the wave equation have also been identified.

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