Minimax State/Residual-Energy Constrained Shapers for Flexible Structures: Linear Programming Approach

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The focus of this paper is on the design of shaped profiles subject to transient state constraints and terminal energy limits. The issue of robustness to modeling errors is addressed by formulating a minimax optimization problem in a linear programming framework, permitting the generation of near-globally optimal shaped profiles. To illustrate the proposed technique, a deflection-limited/residual energy constrained, controller design for flexible structures undergoing rest-to-rest maneuvers will be presented. Techniques for improving robustness by the addition of state sensitivity equations, which increases the robustness in the vicinity of the nominal model, are explored. Results for the benchmark oscillator illustrate the benefit of the proposed approach over traditional designs. [DOI: 10.1115/1.4001275]

1 Introduction

Vibration control of slewing flexible structures has been the subject of research interest in both the aerospace and robotics communities [1,2]. These studies encompass a number of lightweight flexible structures; such applications include large spacecraft and space structures [3], robotic arms [4], gantry cranes [5], hard disk drives [6], etc. In 1957, Smith [1] proposed a wave cancellation technique, termed as "Posicast," to drive a system with one resonant mode to its final position in finite time. Singer and Seering [7] arrived at the same results as did Smith with an input shaping approach. In addition, they proposed a technique for making input shaping commands insensitive to errors in the model parameters, which involved forcing the system's residual energy, and derivative with respect to natural frequency or damping, to zero. Singh and Vadali [2] derived the same results as did Singer and Seering [7] with the design of a time delay prefilter, which provided zeros to the system so as to cancel the poles. Liu and Wie [8] proposed an approach for desensitizing time-optimal control profiles to model uncertainties, which involved decoupling the equations of motion into rigid and flexible body modes. Robustness is achieved by forcing the partial derivative of the decoupled states with respect to the natural frequency to zero at the final maneuver time [8]. Singh and Vadali [2,9] improved the robustness of the shaped input and time-optimal control strategies by applying the pole cancellation technique and requiring the control sequence to satisfy the robustness constraints.

The aforementioned techniques essentially concentrate on eliminating residual vibration and increasing the robustness of input shaping and minimum-time controllers. None, however, address the problem of limiting the large deflection amplitudes normally associated with them. Take for example the system illustrated in Fig. 1; it may be necessary to move the structure a finite distance in the minimum time while limiting the maximum extension/contraction that occurs in the spring. Banerjee et al. [10] proposed the first approach for developing deflection-limited input shaping commands. In their paper, a technique for applying deflection limits at instances where local extrema points occur in the transient deflection is presented. However, the expressions are only derived for control impulse amplitudes of ± 1 warranting the development of bang-off-bang control sequences, which are not time optimal. Robertson and Singhose [11] arrived at the same results as with the aforementioned technique with a discrete approach. Later they presented an approach for developing closed form expressions to the deflection-limiting commands [12]. Their technique incorporated magnitude restricted coasting periods in the preshaped profile. In addition, they developed a robust approach, which involved an extension in the number of profile switch times [13]. Minimizing the worst case performance of the system over the domain of uncertainty was posed as a minimax problem by Singh [14], and Tenne and Singh [15]. The residual energy of the maneuvering system was used as the performance index, and the performance index was sampled over the uncertain domain to determine the maximum. This technique was illustrated for shaping the reference inputs and for determining the robust control profiles.

The preceding papers mainly focus on generating closed form or numerical solutions to the deflection-limiting control problem using nonlinear optimization toolboxes. Fegley et al. [16] presented a linear and quadratic programming formulation for the design of optimal controllers. They present two techniques, which include discretizing the state equations and assuming a finite set of basis functions to represent the state and control profiles. They arrive at convex programming problems, which produce near-globally optimal controller. Sinha and Peng [17] presented a linear programming formulation of the deflection-limited time-optimal control design, which generates near-globally optimal controllers. Both papers [16,17] propose a time weighted penalty function on the terminal errors to determine a minimum-time controller. This formulation asymptotically approaches the minimum-time solution as the penalty weights are increased. However, neither of these papers study the problem of robustness to modeling uncertainties, which is a focus of this paper. The problem statement for state-constrained time-optimal control is outlined in Sec. 2. A linear programming approach for solving a state-constrained time-optimal control problem is developed in Sec. 3. In addition, robustness is achieved by adding sensitivity states to the LP formulation in Sec. 4. Finally, Sec. 5 details a minimax approach with the objective of minimizing the maximum amplitude of transient deflection encountered across a specified domain of model uncertainty.

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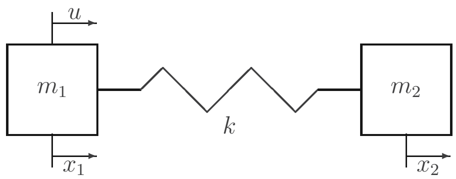

Fig. 1 Undamped floating oscillator

2 Mathematical Formulation

The traditional time-optimal control problem consist of determining the control $u(t)$, which drives the system states x, governed by $\dot{x}(t)=Ax(t)+Bu(t)$ from their specified initial state x_0 to their desired final state x_f, while minimizing the performance index

$$\min J = \int_0^{t_f} dt \tag{1}$$

To guarantee that the input magnitude is physically obtainable and the actuators are capable of performing the desired maneuver, it must also satisfy the constraint

$$u_{\min} \le u(t) \le u_{\max} \tag{2}$$

To ensure that the states are not driven beyond their permissible amplitude, they must additionally satisfy

$$\Delta x_{\min} \le \Delta x(t) = Lx(t) \le \Delta x_{\max} \tag{3}$$

where $\Delta x(t)=Lx(t)$ is a linear combination of states permitting constraints to be imposed on velocities, relative deflection, etc.

For simplicity the control limits are assumed to be symmetric. In addition, for generality purposes, the control input is constrained to $-1 \le u(t) \le 1$, and the states are constrained to $-\delta \le \Delta x(t) \le \delta$, where δ is, for example, the specified deflection or velocity limit. The resulting problem statement for state-limited time-optimal control is

$$\min J = \int_0^{t_f} dt \tag{4a}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4b}$$

$$x(t_0) = x_0 \quad x(t_f) = x_f \tag{4c}$$

$$-1 \le u(t) \le 1 \quad \forall\, t \tag{4d}$$

$$-\delta \le \Delta x(t) \le \delta \quad \forall\, t \tag{4e}$$

where $x(t) \in \mathcal{R}^n$ and $u(t) \in \mathcal{R}^m$. The structure of the time-optimal control profile for such a problem is often difficult to predict, particularly with the addition of state constraints. Nonlinear optimization techniques do not always guarantee global optimality and can be computationally expensive. Linear programming offers an efficient method for handling numerous linear constraints. Although the accuracy is reliant on the number of discrete intervals and convergence tolerance, a Linear Programming (LP) approach is globally convergent and guaranteed to produce a near-globally optimal control strategy irrespective of the system or number of resonant modes being considered [18].

3 Linear Programming Formulation

The problem statement for a standard linear programming problem may be stated as

$$\min c^T z \tag{5a}$$

subject to

$$A_{\text{eq}} z = b_{\text{eq}} \tag{5b}$$

$$Az \le b \tag{5c}$$

$$z_{\text{lb}} \le z \le z_{\text{ub}} \tag{5d}$$

where z, in Eq. (5a), represents a vector of variables to minimize the cost function $c^T z$. First, consider the linearized discrete time state space model with sampling time T_s

$$x(k+1) = Gx(k) + Hu(k) \quad \text{for } k = 1,2, \ldots ,n \tag{6}$$

where G and H are the discrete time state space matrices of the vibratory system. The state response for any input $u(k)$ is

$$x(k+1) = G^k x_0 + \sum_{i=1}^{k} G^{k-i} H u(i) \tag{7}$$

where x_0 corresponds to the initial states of the system. From Eq. (7), the final state response of the system may be represented as

$$x(n+1) = G^n x_0 + \sum_{i=1}^{n} G^{n-i} H u(i) \tag{8}$$

where n is the total number of discrete intervals and $(n+1)T_s$ represents the final maneuver time t_f. Satisfying the equality given in Eq. (8) guarantees that the desired final states are obtained according to the governing dynamics of the system. Rearranging Eq. (8) in the standard equality constraint format for linear problems gives

$$\begin{bmatrix} G^{n-1}H & G^{n-2}H & \ldots & GH & H \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = x_f - G^n x_0 \tag{9}$$

Additionally, state limits are derived from the discrete time state equations. Multiplying Eq. (7) by an output matrix L, defined by $Lx(k)=\Delta x(k)$, gives the expression for state variation at every interval k represented as

$$\Delta x(k+1) = LG^k x_0 + \sum_{i=1}^{k} LG^{k-i} H u(i) \tag{10}$$

For example, assuming $x=[x_1 \;\; x_2 \;\; \dot{x}_1 \;\; \dot{x}_2]^{\mathrm{T}}$ for the system illustrated in Fig. 1, the output matrix, which captures the spring deformation would be given as $L=[1 \;\; -1 \;\; 0 \;\; 0]$. Therefore Eq. (10) must satisfy the inequality

$$-\delta \le LG^k x_0 + \sum_{i=1}^{k} LG^{k-i} H u(i) \le \delta \quad \forall\, k \;\; \in [0 \;\; n] \tag{11}$$

For rest-to-rest maneuvering the initial states x_0 are nominally zero; therefore, Eq. (11) may be rewritten in the standard inequality constraint form

$$\pm \begin{bmatrix} LH & 0 & \ldots & 0 \\ LGH & LH & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ LG^{n-1}H & LG^{n-2}H & \ldots & LH \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \le \delta \tag{12}$$

To guarantee that the control input is obtainable, the saturation constraints are simply given by

$$-1 \le u(k) \le 1 \tag{13}$$

Having specified the final maneuver time, Eqs. (9), (12), and (13) constitute a set of linear equality and inequality constraints, which, in conjunction with assigning c in Eq. (5a) to be a null vector, results in a linear programming problem, which essentially results in a feasible control input. Iterative methods, such as the bisection algorithm, are then used to incrementally converge to the time-optimal solution based on the feasibility of the current maneuver time.

4 Robust Linear Programming

The control profile relies on the model parameters to satisfy the terminal state constraints; therefore, the controller performance is a function of how accurately the system model is in representing the system. The design technique presented in Sec. 3 relied solely on the underlying presumption that no variation in the parameters exist. Since this is never the case, it is desirable to obtain the control strategies, which are insensitive to errors in the estimated parameter values.

In past researches, numerous techniques for desensitizing the control strategy to errors in the model parameters have been developed. Liu and Singh [19] proposed a technique which requires the control input to satisfy boundary conditions on sensitivity states defined as the derivative of the state equations with respect to the uncertain parameters. Robustness is achieved by forcing the sensitivity states to zero at the end of the maneuver. Consider a state space representation of an uncertain linear dynamical system

$$\dot{y} = A(p)y + B(p)u \tag{14}$$

where $y \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, and p is a vector of uncertain parameters. The sensitivity of the states with respect to the ith parameter p_i is

$$\frac{d\dot{y}}{dp_i} = \frac{\partial A}{\partial p_i}y + A\frac{dy}{dp_i} + \frac{\partial B}{\partial p_i}u \tag{15}$$

which can be concatenated to the state equations to create an augmented system model, which can be used for the design of robust controllers. For example, consider the floating oscillator in Fig. 1, whose equations of motion are given as

$$m_1\ddot{x}_1 + kx_1 - kx_2 = u \tag{16a}$$

$$m_2\ddot{x}_2 - kx_1 + kx_2 = 0 \tag{16b}$$

Assuming that the uncertain parameter is the estimated spring stiffness k, the partial derivative of Eq. (16) with respect to k results in

$$\frac{\partial \ddot{x}_1}{\partial k} = \frac{1}{m_1}\left[-x_1 + x_2 - k\left(\frac{\partial x_1}{\partial k} - \frac{\partial x_2}{\partial k}\right)\right] \tag{17a}$$

$$\frac{\partial \ddot{x}_2}{\partial k} = \frac{1}{m_2}\left[x_1 - x_2 + k\left(\frac{\partial x_1}{\partial k} - \frac{\partial x_2}{\partial k}\right)\right] \tag{17b}$$

Clearly the sensitivity expressions featured in Eqs. (17*a*) and (17*b*) are not independent from one another; thus, only one is necessary to capture the dynamics of the two. The relationship between the sensitivity states may be expressed as

$$\frac{\partial \ddot{x}_1}{\partial k} = -\frac{m_2}{m_1}\frac{\partial \ddot{x}_2}{\partial k} \tag{18}$$

To increase the robustness, the following boundary conditions

$$\left.\frac{\partial x_1}{\partial k}\right|_{t_f} = \left.\frac{\partial x_2}{\partial k}\right|_{t_f} = \left.\frac{\partial \dot{x}_1}{\partial k}\right|_{t_f} = \left.\frac{\partial \dot{x}_2}{\partial k}\right|_{t_f} = 0 \tag{19}$$

which correspond to the sensitivity states at the final maneuver time are enforced.

By integrating Eq. (18) twice, backward in time, subject to the boundary conditions presented in Eq. (19), the relationship may be expressed as

$$\frac{\partial x_1}{\partial k} = -\frac{m_2}{m_1}\frac{\partial x_2}{\partial k} \tag{20}$$

Substituting Eq. (20) into Eq. (17) permits the derivation of the general state sensitivity expression of the form

$$\frac{\partial \ddot{x}_2}{\partial k} = \frac{x_1}{m_2} - \frac{x_2}{m_2} - \left(\frac{k}{m_1} + \frac{k}{m_2}\right)\frac{\partial x_2}{\partial k} \tag{21}$$

Equation (21) represents a new equation to be introduced into the system, which desensitizes the system to errors in the spring stiffness k. The equations of motion in the continuous time domain are represented by

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \dfrac{\partial \ddot{x}_2}{\partial k} \end{bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & k & 0 \\ -\dfrac{1}{m_2} & \dfrac{1}{m_2} & \dfrac{k}{m_1} + \dfrac{k}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \dfrac{\partial x_2}{\partial k} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u \tag{22}$$

The discrete time state space equations can be determined from Eq. (22) and used in the same linear programming formulation presented in Sec. 3.

5 Minimax Control Formulation

In the approach described in Sec. 4, controller robustness is achieved by forcing sensitivity states to zero at the final maneuver time. However, since the sensitivity states are evaluated at the nominal system values, the control strategy is generally desensitized to errors near these values. To exploit the knowledge of the domain of uncertainty, it is desirable to develop a control strategy, which is insensitive across the entire domain. In addition, the previous design techniques focused only on achieving robustness at the end of the maneuver, which is an attempt to obtain a more acceptable steady state response. However, transient state-limited control is the focus of this paper; therefore it is desirable to obtain a control action, which desensitizes the transient deflection to errors in the estimated parameter values. The general system model can be represented as

$$M\ddot{x}(t) + C(\alpha)\dot{x}(t) + K(\alpha)x(t) = Du(t) \tag{23}$$

where α represents a finite parametric set of the uncertain parameter that must lie within the domain of uncertainty specified by

$$\alpha^{\mathrm{lb}} \le \alpha \le \alpha^{\mathrm{ub}} \tag{24}$$

where α^{lb} and α^{ub} represent the lower and upper bounds, respectively. Estimates of inertia are often accurate, and hence in this formulation, uncertainties appear in the damping and stiffness matrices only. The goal is to design a control strategy with the objective of minimizing the maximum transient deflection. Therefore the cost function f for each parameter uncertainty may be stated as

$$f^{\alpha} = \max_t |\Delta x^{\alpha}(t)| \tag{25}$$

where the maximum is taken over all admissible time $t \in [0 \;\; t_f]$, and the superscript α designates which uncertain system the deflection term belongs to. The worst (highest) cost can be defined as

$$F = \max_{\alpha} f^{\alpha} \tag{26}$$

Thus the robust problem statement consist of determining the control strategy $u(t)$, which minimizes the maximum level of transient deflection given by

$$\min_{u(t)} F = \min_{u(t)} \max_{\alpha} f^{\alpha} \tag{27}$$

5.1 Minimax Linear Programming. The LP formulation presented in Sec. 3 only considers the nominal plant and uses an iterative approach to arrive at the optimal solution. At each iteration, the final time is specified and a feasibility problem is solved, i.e., the cost function of the LP problem is a vector of zeros multiplying the optimization variables. Time optimality is achieved by reducing the final time or increasing the final time if the problem is feasible or infeasible respectively. To gain robustness with respect to the transient deflection, it is proposed to require the LP formulation to satisfy multiple sets of linear constraints, which reflect the dynamics of each uncertain system. First, the state deflection term is introduced into the cost function of the linear program given by

$$\min c^T z = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ F \end{bmatrix} \tag{28}$$

The new objective function in Eq. (28) requires the optimizer to find the control action, which satisfies all the constraints, while minimizing the maximum level of deflection given by F. It is important to note that the final maneuver time represents a design variable in this formulation, and no sensitivity states are included. The deflection constraint for each system is given by

$$-F \leq \Delta x^{\alpha}(k+1) \leq F \quad \forall\ \alpha \tag{29}$$

where F is constrained to be a positive number.

Recall that the deflection expression may be represented in terms of the discrete time state equation of the form

$$\Delta x^{\alpha}(k+1) = LG(\alpha)^k x_0 + \sum_{i=1}^{k} LG(\alpha)^{k-i} H(\alpha) u(i). \tag{30}$$

Substituting Eq. (30) into Eq. (29) and rearranging into upper and lower inequalities gives

$$-\sum_{i=1}^{k} LG(\alpha)^{k-i} H(\alpha) u(i) - F \leq 0 \quad \forall\ k \in [0\ \ n] \tag{31a}$$

$$\sum_{i=1}^{k} LG(\alpha)^{k-i} H(\alpha) u(i) - F \leq 0 \quad \forall\ k \in [0\ \ n] \tag{31b}$$

where the initial condition x_0 is omitted for rest-to-rest maneuvering. The standard inequality constraint format for linear problems may be represented as

$$-\begin{bmatrix} LH(\alpha) & \ldots & 0 & 1 \\ LG(\alpha)H(\alpha) & \ldots & 0 & 1 \\ \vdots & \ldots & \vdots & 1 \\ LG(\alpha)^{n-1}H(\alpha) & \ldots & LH(\alpha) & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \\ F \end{bmatrix} \leq 0 \tag{32a}$$

$$\begin{bmatrix} LH(\alpha) & \ldots & 0 & -1 \\ LG(\alpha)H(\alpha) & \ldots & 0 & -1 \\ \vdots & \ldots & \vdots & -1 \\ LG(\alpha)^{n-1}H(\alpha) & \ldots & LH(\alpha) & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \\ F \end{bmatrix} \leq 0 \tag{32b}$$

The upper and lower inequality matrices of Eqs. (32a) and (32b) may then be combined into one global matrix of $\mathcal{R}^{2n \times n+1}$ dimension, given by

$$Q^{\alpha} z \leq 0 \tag{33}$$

The terminal constraints for the discrete time state space equations, given by Eq. (8), must also be satisfied. However, each uncertain parameter $\alpha \in [\alpha^{\mathrm{lb}}\ \ \alpha^{\mathrm{ub}}]$ presents a system with slightly different governing dynamics. Therefore the linear program cannot satisfy each set of equality constraints simultaneously. The only feasible approach is to pose them as inequality constraints and require the solution to satisfy the terminal constraints of each system within a specified tolerance. Essentially this provides slack to the terminal position and velocity states. Therefore it is practical to invoke the tolerance on the residual energy of each system given by

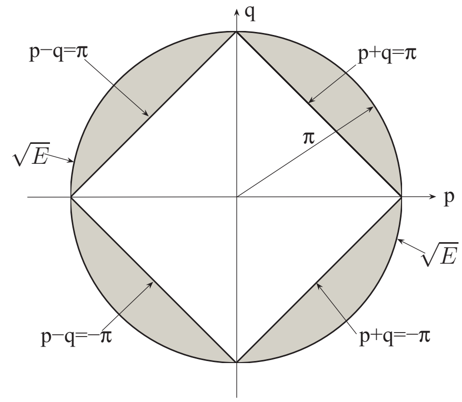

Fig. 2 Linearly approximated residual energy constraints

$$E(\alpha) = \frac{1}{2}\dot{x}^{\alpha T} M \dot{x}^{\alpha} + \frac{1}{2}[x^{\alpha} - x_{\mathrm{ref}}]^T K(\alpha)[x^{\alpha} - x_{\mathrm{ref}}] \tag{34}$$

which is normally referred to as the pseudo energy equation since it is associated with a hypothetical spring, whose potential energy is zero when $x = x_{\mathrm{ref}}$, which is the desired final displacement state. Equation (34) is a quadratic function of the system states and cannot be incorporated into a linear programming problem. Equation (34) represents a nth dimensional hypersphere, and if one can bound it with hyperplanes, one can pose the constraint in a linear form. Consider the expanded form of Eq. (34) rewritten as

$$E(\alpha) = \left[\frac{1}{\sqrt{2}}\sqrt{M}\dot{x}^{\alpha}\right]^T \left[\frac{1}{\sqrt{2}}\sqrt{M}\dot{x}^{\alpha}\right] + \left[\frac{1}{\sqrt{2}}\sqrt{K(\alpha)}(x^{\alpha} - x_{\mathrm{ref}})\right]^T \left[\frac{1}{\sqrt{2}}\sqrt{K(\alpha)}(x^{\alpha} - x_{\mathrm{ref}})\right] \tag{35}$$

Since the residual energy is a scalar quadratic term, where the kinetic energy is only a function of the velocity states and potential energy is only a function of the position states, bounding the residual energy in state space can be represented as requiring the states to lie in a hypersphere, whose radius is the residual energy bound. By circumscribing or inscribing the hypersphere with bounding hyperplane, Eq. (35) can be approximated as

$$-\pi \leq \frac{1}{\sqrt{2}}\sqrt{M}\dot{x}^{\alpha} + \frac{1}{\sqrt{2}}\sqrt{K(\alpha)}(x^{\alpha} - x_{\mathrm{ref}}) \leq \pi \tag{36}$$

where π is a vector, which represents a user specified design variable constraining the square root of the system's terminal potential and kinetic energy (Fig. 2 illustrates the linear approximation of a circle in two dimensional space). Equation (36) however, consists of all the system states; therefore, a nth order system needs to be constrained on n axes. Additionally, the energy terms are no longer restricted to be positive in sign; therefore, they must also be constrained along the negative axis. "In geometry of n dimensions, an orthant is one of the 2^n parts of Euclidean space defined by constraining each Cartesian coordinate axis to be positive or negative. That is, an orthant is the analogue of a quadrant in the plane, and is defined by a system of inequalities $\epsilon_i x_i \geq 0$ for $1 \leq i \leq n$ on the coordinates x_i, where $\epsilon_i = \pm 1$" [20]. Therefore, the fourth order system in Fig. 1 would require a minimum of 16 constraint equations to make a reasonable linear approximation. Since the error in approximating an orthant of a unit hypersphere with a hyperplane is given by $1-(1/\sqrt{n})$ [21], where n is the dimension of the system, it is clear that the approximation error increases as a function of the system order. By decoupling the system states into the rigid and flexible body equations of motion, a more accurate approach can be fashioned. Since individual energy expressions may be written for each of the decoupled states, they are always second order and are therefore easily constrained in 2D space. The decoupled state vector, assuming a Rayleigh damping matrix, defined by $y=[\theta \; q_1 \; \ldots \; q_n]$, may be written in the form

$$\begin{bmatrix} \ddot{\theta}^\alpha \\ \ddot{q}_1^\alpha \\ \vdots \\ \ddot{q}_n^\alpha \end{bmatrix} + \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 2\zeta_1\omega_1(\alpha) & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 2\zeta_1\omega_n(\alpha) \end{bmatrix} \begin{bmatrix} \dot{\theta}^\alpha \\ \dot{q}_1^\alpha \\ \vdots \\ \dot{q}_n^\alpha \end{bmatrix} + \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & \omega_1^2(\alpha) & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \omega_n^2(\alpha) \end{bmatrix} \begin{bmatrix} \theta^\alpha \\ q_1^\alpha \\ \vdots \\ q_n^\alpha \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} u(t) \quad (37)$$

where θ represents the rigid body state associated with system mass and q_i represents the flexible body states, each associated with a natural frequency and damping ratio. The relationship between the physical and decoupled states is given by

$$x = Vy \quad (38)$$

where V represents the similarity transformation matrix determined from the eigenvectors of the system. Therefore the boundary conditions which accompany the decoupled states are found by

$$y(t_0) = V^{-1}x_0 \quad (39a)$$

$$y(t_f) = V^{-1}x_f \quad (39b)$$

Rewriting the decoupled state space presented in Eq. (37) leads to the following set of differential equations of motion

$$\ddot{\theta}^\alpha = \phi_0 u(t) \quad (40a)$$

$$\ddot{q}_i^\alpha + 2\zeta_i\omega_i(\alpha)\dot{q}_i^\alpha + \omega_i^2(\alpha)q_i^\alpha = \phi_i u(t) \quad \text{for } i=1,\ldots,n \quad (40b)$$

Thus the residual energy expressions for the rigid and flexible body modes may be represented by

$$E^{\text{rb}} = \frac{1}{2}[\dot{\theta}^\alpha]^2 \quad (41a)$$

$$E_i^{\text{fb}} = \frac{1}{2}[\dot{q}_i^\alpha]^2 + \frac{1}{2}\omega_i^2(\alpha)[q_i^\alpha]^2 \quad \text{for } i=1,\ldots,n \quad (41b)$$

Clearly the rigid body mode only contains kinetic energy, due to the rigid body equation of motion being an explicit function of the system mass. Consequently, applying an inequality constraint to Eq. (41a) results in no penalty to the terminal position of the rigid body state. To address this, a pseudo potential energy term is added to the rigid body energy expression. This acts as a hypothetical spring, whose potential energy is zero when $\theta=\theta_{\text{ref}}$, penalizing the terminal state. θ_{ref} is the desired final displacement of the rigid body mode in the modal space. The new residual energy expression is given by

$$E^{\text{rb}} = \frac{1}{2}[\dot{\theta}^\alpha]^2 + \frac{1}{2}k_p[\theta^\alpha - \theta_{\text{ref}}]^2 \quad (42a)$$

$$E_i^{\text{fb}} = \frac{1}{2}[\dot{q}_i^\alpha]^2 + \frac{1}{2}\omega_i^2(\alpha)[q_i^\alpha]^2 \quad \text{for } i=1,\ldots,n \quad (42b)$$

where k_p is an arbitrarily selected positive value since it is associated with a pseudo spring constant. The expressions in Eq. (42) are again quadratic functions of the system states and cannot be incorporated into a linear problem. However, similar to the approach of Eq. (34), Eq. (42) may be expanded and linearly approximated to satisfy the inequality

$$-\pi \le \frac{1}{\sqrt{2}}\dot{\theta}^\alpha \pm \frac{1}{\sqrt{2}}k_p^{1/2}[\theta^\alpha - \theta_{\text{ref}}] \le \pi \quad (43a)$$

$$-\pi \le \frac{1}{\sqrt{2}}\dot{q}_i^\alpha \pm \frac{1}{\sqrt{2}}\omega_i(\alpha)q_i^\alpha \le \pi \quad (43b)$$

Satisfying $\pm$ of each term in Eq. (43) accounts for all the necessary conditions to fully constrain each decoupled energy expression. Defining q to be the square root of the kinetic energy of the mode and p as the square root of the potential energy of the mode, Fig. 2 illustrates the approximation of the circle corresponding to the residual energy of the rigid/flexible model (Eq. (42)), with a polygon given by Eq. (43), where the shaded area represents the approximation error. Equation (43) may now be rewritten in the form

$$-\pi \le P(\alpha)[y^\alpha - \bar{y}_f] \le \pi \quad (44)$$

where y is the decoupled state vector given by

$$y = \begin{bmatrix} \theta & \dot{\theta} & q_1 & \dot{q}_1 & \ldots & q_n & \dot{q}_n \end{bmatrix}^{\text{T}} \quad (45)$$

and the matrix $P(\alpha)$ is given by

$$P(\alpha) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{k_p} & 1 & 0 & \ldots & \ldots & 0 \\ -\sqrt{k_p} & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \omega_1(\alpha) & 1 & \vdots & \vdots \\ \vdots & \vdots & -\omega_1(\alpha) & 1 & \vdots & \vdots \\ \vdots & \vdots & \ldots & \ldots & \omega_n(\alpha) & 1 \\ 0 & 0 & \ldots & \ldots & -\omega_n(\alpha) & 1 \end{bmatrix} \quad (46)$$

The final inequality constraint on the residual energy term is represented by

$$-\pi + P(\alpha)y_f \le P(\alpha)y^\alpha \le \pi + P(\alpha)y_f \quad (47)$$

where y^α is evaluated at the final maneuver time. Recall that the final state response of the system in terms of the discrete time state equation may be represented as

$$y^\alpha(n+1) = G(\alpha)^n y_0 + \sum_{i=1}^{n} G(\alpha)^{n-i}H(\alpha)u(i) \quad (48)$$

Substituting Eq. (48) into Eq. (47) and rearranging into upper and lower inequalities gives

$$-\sum_{i=1}^{n} P(\alpha)G(\alpha)^{n-i}H(\alpha)u(i) \le \pi - P(\alpha)y_f \quad (49a)$$

$$\sum_{i=1}^{n} P(\alpha)G(\alpha)^{n-i}H(\alpha)u(i) \le \pi + P(\alpha)y_f \quad (49b)$$

where the initial condition is again removed for rest-to-rest maneuvering. Equation (49) may be expanded into standard inequality form for a linear programming problem as

$$-\begin{bmatrix} P(\alpha)G(\alpha)^{n-1}H(\alpha) & \ldots & 0 \end{bmatrix} \begin{bmatrix} u \\ F \end{bmatrix} \le \pi - P(\alpha)y_f \quad (50a)$$

$$\begin{bmatrix} P(\alpha)G(\alpha)^{n-1}H(\alpha) & \ldots & 0 \end{bmatrix} \begin{bmatrix} u \\ F \end{bmatrix} \le \pi + P(\alpha)y_f \quad (50b)$$

Similar to the deflection inequality, the upper and lower matrices in Eq. (50) may be combined into a global matrix of dimension $\mathcal{R}^{2r \times n+1}$, where r refers to the total number of states in the system, represented as

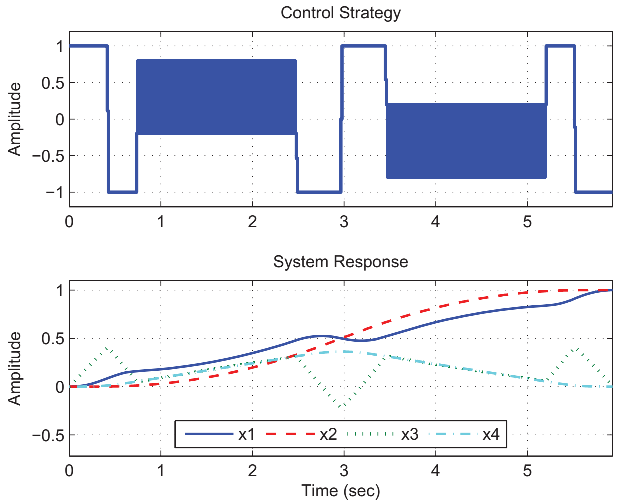

Fig. 3 Nonrobust deflection-limited control profile for δ=0.15

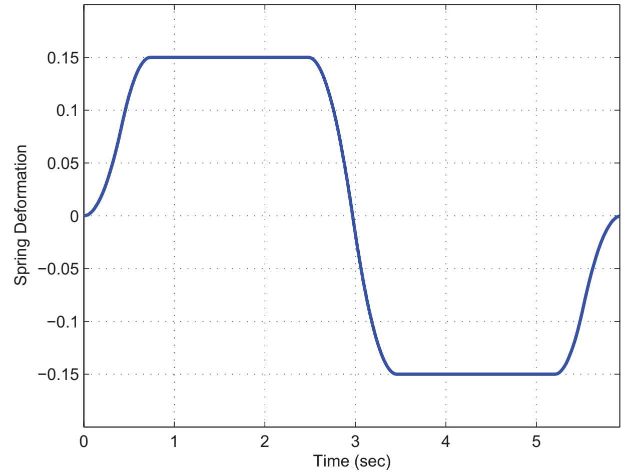

Fig. 4 Deflection profile

$$R^{\alpha} z \leq \pi \pm P(\alpha) y_f \tag{51}$$

The resulting problem statement for the deflection-limited minimax formulation is given by

$$\min c^T z = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ F \end{bmatrix} \tag{52a}$$

$$\text{s.t.} \quad \left. \begin{array}{l} Q^{\alpha} z \leq 0 \\ R^{\alpha} z \leq \pi \pm P(\alpha) y_f \\ -1 \leq u(k) \leq 1 \end{array} \right\} \quad \forall\ \alpha \in [\alpha^{\text{lb}} \ \ \alpha^{\text{ub}}] \tag{52b}$$

Note that the residual energy constraints used in this formulation are functions of both the rigid body and flexible modes unlike the metric used in the design of extra insensitive shapers [22], which use the contributions of the flexible modes in the determination of the optimal input shaper. When no rigid body exists in the system, the two constraints are identical.

6 Numerical Simulations

In this section, the proposed techniques are illustrated on the undamped floating oscillator problem (Fig. 1) undergoing a rest-to-rest maneuver, whose equations of motion are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{53}$$

with the boundary conditions

$$\underset{\sim}{x}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dot{\underset{\sim}{x}}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \underset{\sim}{x}(t_f) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \dot{\underset{\sim}{x}}(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{54}$$

First, the linear program is solved using the nonrobust formulation presented in Sec. 3, which minimizes the final time, requires satisfaction of the terminal state constraints and transient deflection constraint, and only considers the nominal model. The nominal system parameters for all the simulation are selected as $k=m_1=m_2=1$, and the permissible deflection level is specified as $\delta=0.15$. Figure 3 illustrates the control profile and the nominal system response with a total maneuver time of $t_f=5.9287$ s.

The figure clearly shows that when the deflection limit is reached, the system enters a period with identical mass velocities and similar displacements in the system response. We will refer to this as coasting. Figure 4 illustrates the evolution of the deflection, which includes two coasting periods. However, in the coasting period, the control profile exhibits a chattering behavior. In contrast to the analytical solution, the switch time locations of the discrete profile are limited to occurring at prescribed intervals; thus, it is impossible to synchronize the velocities exactly. The solution is a rapidly switching input sequence, which causes the state velocities to alternate back and forth between each interval. This behavior, in turn, sustains a constant state deflection value for the duration of the coast. However, Robertson and Singhose [12] evidenced the coasting period amplitude of the analytical solution to be a constant input magnitude of $u(t)=k[1+(m_1/m_2)]\delta$. Therefore, to ameliorate the chattering, the control profile is post-processed by forcing the input magnitude to be constant at locations where the deflection constraint is active. Figure 5 illustrates the modified profile along with the residual energy distribution for the pre- and post-processed control strategies. This clearly demonstrates that the modification may be achieved with little or no alteration on the simulated response.

Next, state sensitivity equations are added to the formulation, and the linear program is solved using the robust approach in Sec. 4. Figure 6 illustrates the robust control profile along with the nominal system response. The overall maneuver time of $t_f=6.5924$ s is 0.6637 s greater than the nonrobust; thus, robustness is not obtained without some penalty. Additionally, the added sensitivity states require two additional switch times in the control profile.

In the final example, the linear program is solved using the minimax approach, as outlined in Sec. 5, and five evenly distributed stiffness parameters, which represent the dynamics of the lower, upper, middle, and nominal valued systems. The domain of

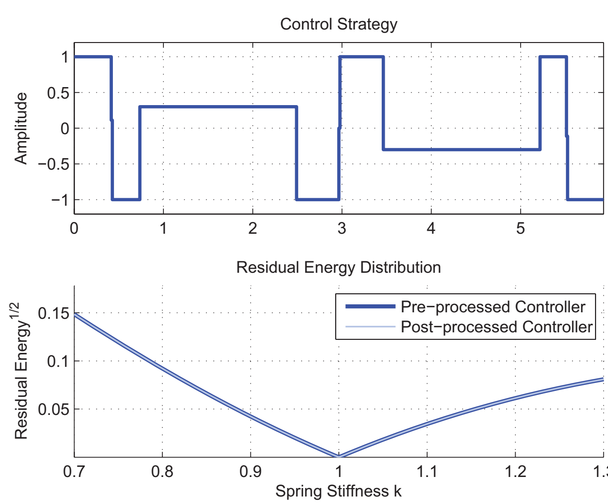

Fig. 5 Post-processed control strategy

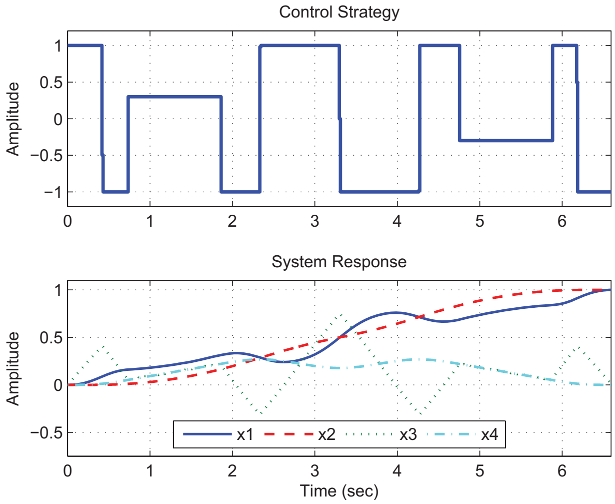

Fig. 6 Robust deflection-limited control profile for $\delta=0.15$

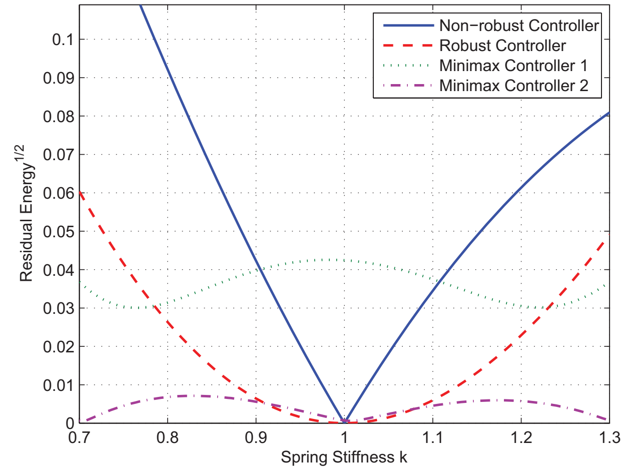

Fig. 8 Residual energy distribution

uncertainty is specified as $0.7 \leq k \leq 1.3$, with a residual energy tolerance of $\pi=0.03$ for constraints given by Eq. (43) and a pseudo spring constant of $k_p=0.7$. The decoupled state space equations are found to be

$$\begin{bmatrix} \ddot{\theta}(t) \\ \ddot{q}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \omega^2(\alpha) \end{bmatrix} \begin{bmatrix} \theta(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{m_1+m_2} \\ \dfrac{-1}{m_1+m_2} \end{bmatrix} u(t) \tag{55}$$

where the state transformation and natural frequency are given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -\dfrac{m_2}{m_1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \theta(t) \\ q(t) \end{bmatrix} \omega(\alpha) = \sqrt{\frac{k(\alpha)(m_1+m_2)}{m_1 m_2}} \tag{56}$$

and boundary conditions are given as

$$\begin{gathered} \theta(t_0) = \dot{\theta}(t_0) = 0, \quad \theta(t_f) = x_1(t_f), \quad \dot{\theta}(t_f) = 0 \\ q(t_0) = \dot{q}(t_0) = 0, \quad q(t_f) = \dot{q}(t_f) = 0 \end{gathered} \tag{57}$$

To conduct a fair comparison, the same final time determined in the robust approach is used here. The cost function was determined to be $F=0.18$, which occurred at $k=0.7$. Due to the maximum deflection occurring at $k=0.7$, the control profile is driven by the uncertainty; thus, the amplitude of the coasting periods is also driven by this uncertainty. Therefore, the control profile, as illustrated in Fig. 7, is post-processed according to $u(t)=k[1+(m_1/m_2)]F$. Figure 7 illustrates the resulting profile along with the simulation results for the system with a stiffness of 0.7, which corresponds to the plant with the largest deflection.

Figures 8 and 9 illustrate the variation in the residual energy and maximum deflection with respect to the uncertain stiffness parameter k. Clearly the robust solution is predominately superior to the nonrobust solution, as evidenced by the substantial increase in residual vibration for trivial perturbations near the nominal parameter value. However, as shown in Fig. 9, both exhibit deflection distributions that are very sensitive to parameter variations. The minimax solution, however, remains equally desensitized in residual energy and is lower than either of the domain limits. The maximum deflection is slightly higher at the nominal value, though it is significantly desensitized across the entire domain of uncertainty. Since the overall energy distribution appears high, a second minimax problem is solved with a 20% extended maneuver time and residual energy tolerance of $\pi=0.0005$. Clearly the results for residual energy and maximum deflection distribution are far superior to any of the preceding controller designs, as shown by the dash-dot line in Figs. 8 and 9. Table 1 summarizes the performance of each controller and reports a percent change

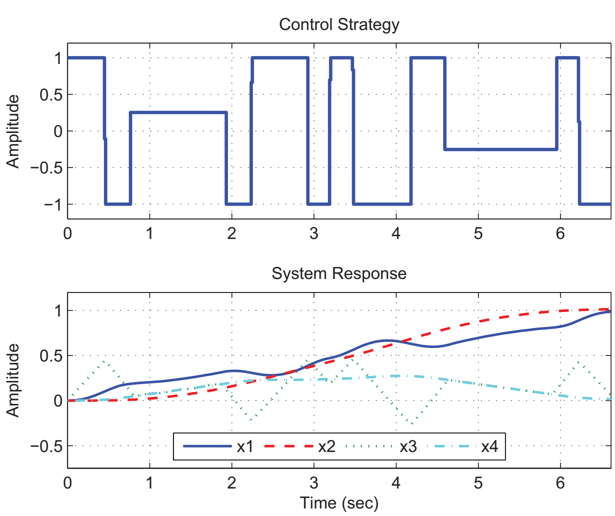

Fig. 7 3-Plant deflection-minimized energy-limited control profile

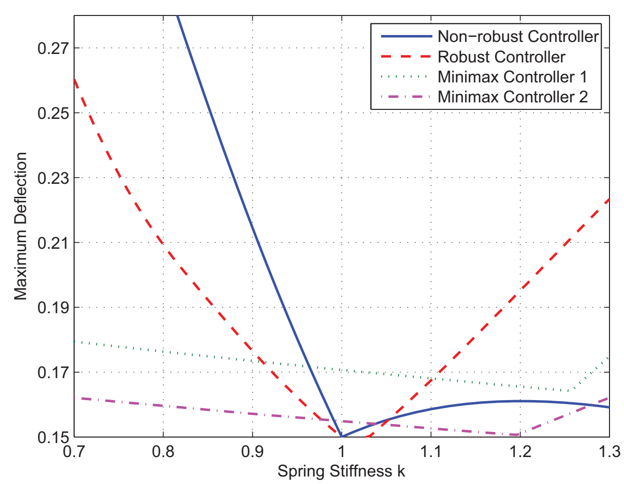

Fig. 9 Maximum deflection distribution

Table 1 Controller performance summary

Performance term	Nonrobust	Robust (%)	Minimax 1 (%)	Minimax 2 (%)
Final time (t_f)	5.9287	111.2	111.2	133.4
Maximum $\sqrt{E}$	0.1485	59.4	71.3	95.2
Maximum $\Delta x(t)$	0.3809	31.7	52.7	57.4

from the nonrobust solution. It is clear that there is an increase in the maneuver time, which trades off significant reduction in the residual energy and maximum deflection.

7 Conclusion

This paper addresses the problem of designing time-optimal control strategies, which limit the maximum magnitude of transient deflection experienced in a maneuvering flexible structure. Linear programming techniques are developed to determine robust and nonrobust control strategies for the deflection-limiting problem. The linear programming solution can contain rapid chattering of the control, which can be post-processed to derive a control profile with few switches. In addition, a minimax approach, with the goal of minimizing the maximum transient deflection that occurs over a range of parameter uncertainty, is presented. The proposed techniques are illustrated on a benchmark floating oscillator problem. The residual energy and maximum deflection distribution for each controller are studied across a specified domain of uncertainty in the spring stiffness. The results demonstrate that the robust and nonrobust solutions produce undesirable magnitude of transient deflection for minor perturbations in the model parameter. However, for an acceptable level of residual vibration, the minimax solution can achieve highly desensitized transient deflection across the entire domain of uncertainty. Furthermore, the formulations and illustrations presented in this paper place emphasis on system deflection; however, these methods are equally applicable to other transient state limitations such as velocity constraints. Lastly, the minimax approach formulated in this paper illustrates only one approach to a larger class of problems. The method essentially consist of three independent variables: final maneuver time t_f, residual energy tolerance π, and maximum deflection coefficient F. The formulation requires two to be specified, while the third is minimized. Alternatively, one could delineate a maximum deflection limit F for all uncertain systems while minimizing the maximum residual energy π, via a minimax formulation. Conversely, for defined limits F and π, an iterative algorithm could be implemented to minimize the final maneuver time t_f.

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