Parametric Control of Combustion Thermo-Acoustic Instabilities

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***Abstract*—In this brief, we present a parametric controller design for thermo-acoustic combustion instabilities in a propulsion system. A generalized wave equation describes the dynamic behavior of second-order nonlinear oscillations. The control algorithm is developed based on the first two modes using the time-averaged simplification of the governing coupled nonlinear system equations. The closed-loop control law design is based on finding an optimal phase angle such that the heat release produced by secondary oscillatory fuel injection is out of phase with the mode's pressure oscillations, and on finding the limits on the controller gain, to ensure the system stability. These gains are then optimized using three different performance indices, namely, integral of the time absolute error (ITAE), integral of the absolute error (IAE), and finally, integral of square of the error (ISE). Simulations illustrate that the proposed techniques are able to effectively attenuate the unwanted oscillations.**

***Index Terms*—Combustion instabilities control, Galerkin approximation, harmonic control, nonlinear, parametric controller, wave equation.**

I. Introduction

Continuous combustion processes are prominently encountered in many applications related to power generation, rockets, propulsion systems, and many other industrial applications. Examples include industrial burners, steam and gas turbines, waste incinerators, jet and ramjet engines, and rocket engines.

Combustion of reactants in a confined volume favors excitation of unsteady motions over a broad range of frequencies. A relatively small portion of the released energy will produce both random fluctuations or noise and, under many circumstances, organized oscillation, generically known as *combustion instabilities* [1]. Owing to the high energy densities and low viscous losses in the combustion chambers, the likelihood of combustion instabilities is high. The accompanying heat transfer, and structural vibration are often unacceptable, causing malfunction or perhaps, in extreme cases, failure of the system.

Generally speaking, there are two general strategies to follow in treating instabilities in any mechanical system or process: changing the design of the system which involves hardware modifications and is generally referred to as *passive control*, or by introducing some form of control that involves actuators which modify the system dynamics, this approach is known as *active control*.

Among the standard practices in passive control of combustion instabilities are the use of baffles, resonating cavities, and acoustic liners. The most important drawback is that a given design is effective only over a fairly narrow frequency range and operating conditions.

Active control provides an attractive alternative to conventional passive approaches without modifying the physical design. Modulation of primary fuel supply, airflow, or introducing a secondary fuel supply are examples of combustion process parameters that can be changed to achieve the control objectives.

Fung *et al.* [1] were the first to develop a control-oriented theoretical representation of liquid-fueled propulsion system. Their model has been used in many journal publications [1]–[4]. In this brief, we exploit Fung's theoretical model to design a parametric controller.

The proposed control strategy is based on the concept of *harmonic control input* to eliminate the harmonic thermo-acoustic instabilities. This concept has received great attention in many experimental works reported in [5]–[7].

The dynamics of combustion process is represented as a parametrically excited system with a harmonic input and the energy based approach is used for synthesizing the controller. The per cycle energy production of the control input is analytically derived and the input is selected to minimize the energy production of the control input per cycle.

The brief is organized as follows. In Section II, the combustion process theoretical model developed by Culick, *et al.* is presented, no attempt is made to rederive these equations, further details can be found in [1]. In Section III, the general class of parametric systems is briefly reviewed, and the parametric controller design is presented, followed by determination of the constraints of the controller gain which ensure the stability of the system. In Section IV, the controller gains are optimized using integral of the time absolute error (ITAE), integral of the absolute error (IAE), and integral of square of the error (ISE) performance indices, followed by simulations, brief discussion and concluding remarks.

II. Theoretical Combustion Model

In this section, a theoretical model for a two-phase flow in a liquid-fueled propulsion system is presented. The final form of the conservation equations are [1]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}_g) = \mathcal{W} \tag{1}$$

$$\rho \frac{\partial \vec{u}_g}{\partial t} + \rho \vec{u}_g \cdot \nabla \vec{u}_g + \nabla p = \vec{\mathcal{F}} \tag{2}$$

$$\frac{\partial p}{\partial t} + \vec{u}_g \cdot \nabla p + \bar{\gamma} p \nabla \cdot \vec{u}_g = \mathcal{P}. \tag{3}$$

The source terms in these equations represent the exchange of mass, momentum and energy between liquid and gas medium

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in the chamber. The governing wave equation is derived by expressing all dependent variables in (1)–(3) as the sum of mean and fluctuating components [1]

$$\nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = h + h_c \tag{4}$$

$$\vec{n} \cdot \nabla p\prime = -\left(\vec{f} + \vec{f_c}\right) \tag{5}$$

where, p' is the pressure fluctuation, $\bar{a}$ is the average speed of sound in the chamber, and where $\vec{n}$ is a unit vector normal to the surface. Quantities h and $\vec{f}$ accommodate the influences of mean flow, combustion, and acoustic motions. Terms h_c and $\vec{f_c}$ represent the effects of external control input. In any active control scheme appropriate external influence must be applied to the system. In case of combustion instabilities this term must be treated as source terms in the conservation equations of the system under consideration. Terms h_c and f_c account, in general, for the control inputs [1]–[4]. Their definition depends on the control method, in the proposed scheme, the controller is based on modulation of a secondary fuel supply derived by [1], therefore, $h_c \equiv u(t)$.

If the right-hand sides of ((4) and (5)) are treated as small perturbations of classical acoustics, the solution can be approximated using the separation of variables as a function of normal modes ψ_n, with time varying amplitude η_n, consequently, the unsteady pressure can be expressed as [1], [3]

$$p'(\mathbf{r}, t) = \bar{p}\sum_{n=1}^{\infty} \eta_n(t)\psi_n(\mathbf{r}) \tag{6}$$

where ψ_n is the eigenfunction satisfying

$$\left.\begin{aligned}\left(\boldsymbol{\nabla}^2 + \mathbf{k_n^2}\right)\boldsymbol{\psi}_\mathbf{n}(\vec{\mathbf{r}}) = \mathbf{0}\\ \vec{n}\cdot\nabla p' = 0\end{aligned}\right\}. \tag{7}$$

For pure longitudinal oscillations in a uniform chamber, ψ_n is given by

$$\psi_n(x) = \cos\frac{n\pi x}{L}. \tag{8}$$

The set of equivalent ordinary differential equations with the unknown time varying amplitude $\eta_n(t)$ for each mode can be obtained via Galerkin's method

$$\ddot{\eta}_n + \omega_n^2\eta_n = u(t) \\ \underbrace{-\sum_{i=1}^{\infty}[D_{ni}\dot{\eta}_i + E_{ni}\eta_i] - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}[A_{nij}\dot{\eta}_i\dot{\eta}_j + B_{nij}\eta_i\eta_j]}_{F_n}. \tag{9}$$

Expressions for the constants A, B, D, E can be found in [8], the numerical values are taken from [1].

III. Method of Time Averaging

The progression from partial differential equations (4), (5) to the set of ordinary differential equations (9) is a great simplification for the analysis. The intent here is to reduce the second-order equations to a set of first-order equations using the time averaging method [9]. The basis for this approach is the fact that many of the observed instabilities are essentially periodic with amplitudes slowly changing in time [1], [10].

The set of ordinary differential equations (9) represents the behavior of a forced oscillator for which, the amplitude η, its rate of change $\dot{\eta}$, and the total mechanical energy $\mathcal{E}_n$ of each mode are, respectively

$$\begin{aligned}\eta_n &= R_n(t)\sin(\omega_n t + \phi_n)\\ &= A_n(t)\sin\omega_n t + B_n(t)\cos\omega_n t \qquad (10)\\ \dot{\eta}_n &= \omega_n R_n\cos(\omega_n t + \phi_n)\\ &\quad + \underbrace{\left[\dot{\phi}_n R_n\cos(\omega_n t + \phi_n) + \dot{R}_n\sin(\omega_n t + \phi_n)\right]}_{\approx 0\ (time\ averaging\ assumption)} \qquad (11)\\ &= \omega_n\left(A_n(t)\cos(\omega_n t) - B_n(t)\sin(\omega_n t)\right) \qquad (12)\\ \mathcal{E}_n &= \frac{1}{2}\omega_n^2\eta_n^2 + \frac{1}{2}\dot{\eta}_n^2. \qquad (13)\end{aligned}$$

Conservation of energy of the averaged motion requires that the rate of change of time-averaged energy of the oscillator equal the time-averaged rate of work done [1], [11], i.e.

$$\left.\begin{aligned}\langle\mathcal{E}_n\rangle &= \tfrac{1}{\tau}\textstyle\int_t^{t+\tau}\mathcal{E}_n d\acute{t}\\ \langle\dot{\eta}_n F_n\rangle &= \tfrac{1}{\tau}\textstyle\int_t^{t+\tau}\dot{\eta}_n F_n d\acute{t}\\ \tfrac{d}{dt}\langle\mathcal{E}_n\rangle &= \langle\dot{\eta}_n F_n\rangle\end{aligned}\right\} \tag{14}$$

where F_n is defined in (9), $\langle \bullet \rangle$ is an averaged quantity, and τ is defined as the interval of averaging that can be reasonably taken as the period of the fundamental mode for the pure longitudinal oscillations considered here [10]. Assuming that $\dot{R}_n(t) \ll 1$, and $\dot{\phi}_n(t) \ll 2\pi$, we can set the quantity in the square brackets in (11) to zero and take R_n and ϕ_n to be constants when integrals (14) are calculated [1], [3], [9]

$$\left.\begin{aligned}\dot{A}_n &= \tfrac{1}{\omega_n\tau}\textstyle\int_t^{t+\tau} F_n\cos\omega_n t' dt'\\ \dot{B}_n &= \tfrac{-1}{\omega_n\tau}\textstyle\int_t^{t+\tau} F_n\sin\omega_n t' dt'\end{aligned}\right\}. \tag{15}$$

The homogeneous solutions of the linear part $\dot{A}_n$ and $\dot{B}_n$ can be obtained as [11]

$$A_n = z_n e^{\alpha_n t}\cos(\theta_n t - \xi_n) \tag{16}$$

$$B_n = -z_n e^{\alpha_n t}\sin(\theta_n t - \xi_n) \tag{17}$$

where z_n and ξ_n are the integration constants. Substitution of the time averaged quantities A_n and B_n in the mode amplitude and its rate of change given in ((10) and (11)) yields

$$\eta_n(t) = z_n e^{\alpha_n t}\sin\left((\omega_n - \theta_n)t + \xi_n\right) \tag{18}$$

$$\dot{\eta}_n(t) = z_n\omega_n e^{\alpha_n t}\cos\left((\omega_n - \theta_n)t + \xi_n\right). \tag{19}$$

The coefficient α_n and θ_n correspond, respectively, to the nth mode linear growth rate, and the shift from classical acoustic frequency ω_n.

IV. Parametric Controller Design

Parametric excitations are descriptive of cases where the excitation appears as a time varying modification of the system parameter. The phenomenon of parametric vibration occurs frequently in various fields of science and engineering and is dealt with in different textbooks, e.g., Nayfeh and Mook (1979), Cartmell (1990). Rayleigh, was perhaps the first scientist to provide a theoretical basis of sound generation by heat addition. Ever since, parametric resonance and stability has been studied in connection with surface waves, string vibrations, Euler columns, electric circuits and vortex shedding vibrations. A general class of parametric systems can be represented by [12]

$$\ddot{x} + f(x,\dot{x}) + g(x) + v(t)x = 0 \tag{20}$$

where $f(x,\dot{x})$ is the nonlinear damping, $g(x)$ is the nonlinear stiffness and $v(t)x$ is the parametric excitation which can be of a higher order. This equation describes the vibration of a general class of nonlinear systems subject to a linear parametric excitation. If the term $v(t)$ can be altered in a real system, then it becomes the control source of the equation. Therefore, selecting $v(t)$ to be harmonic ensures system stability if its phase and frequency are properly tuned provided that $\dot{x}f(x,\dot{x}) \geq 0$ and $xg(x) > 0$ [9], [12].

Since the oscillation given in (18) is harmonic of frequency $\Omega_n = \omega_n - \theta_n$, we can assume a sinusoidal control input. Therefore, the form of $v(t)$ in (20) can be taken as

$$v_n(t) = a_n \sin(\ell\Omega_n t + \phi_n + \xi_n) \tag{21}$$

where a_n is the controller gain, ℓ is an integer. The control law $u(t)$ is assumed to be [12]

$$u(t) = b_n \sum_{i=1}^{n} \left(-\frac{v_i(t)}{b_i}\eta_i(t)\right) \tag{22}$$

where b_n are the elements of the control influence matrix defined as

$$b_n = \frac{\bar{a}}{\bar{p}E_n^2}\psi_n(x_a) \tag{23}$$

where x_a is the point actuator location, substituting (22) in (9) we have

$$\ddot{\eta}_n + \omega_n^2\eta_n = -F_n + b_n \sum_{i=1}^{m} \left(-\frac{v_i(t)}{b_i}\eta_i(t)\right) \tag{24}$$

where m is the number of modes used to approximate the systems dynamics. It is well known that the parametric resonance occurs most readily when the frequency of the parametric excitation is twice as that of the oscillation to be controlled, hence, in (21), $\ell = 2$ seems to be a logical choice.

Theoretically, the unsteady pressure in the combustion chamber requires an infinite number of modes to completely describe its behavior. In practice, however, the infinite series can be truncated at a certain level large enough but finite. The synthesis of the controller in this brief is based on a 4-mode system, the lower two modes are controlled, and the higher two modes are considered to be residual [1].

To avoid the cumbersome algebra involved in the controller synthesis, numerical values of the uncontrolled system growth coefficients and frequency shifts for the first and the second modes are given in Table I [13]. For these parameters, ((18), (19) and (21)) become

$$\left.\begin{aligned} \eta_n &= z_n e^{\alpha_n t}\sin(\Omega_n t + \xi_n) \\ \dot{\eta}_n &= z_n\omega_n e^{\alpha_n t}\cos(\Omega_n t + \xi_n) \\ v_n &= a_n \sin(2\Omega_n t + \phi_n + \xi_n) \end{aligned}\right\}. \tag{25}$$

Now, let us consider the per cycle energy production, E_{ni}, due to $v_n\eta_n$ terms

$$E_{ni} = \int_0^{\frac{2\pi}{\Omega_n}} -\frac{b_n}{b_i}v_n(t)\eta_i(t)\dot{\eta}_n(t)dt \quad n,i = 1\ldots 2. \tag{26}$$

The energy produced by the control input is given by

$$\varepsilon_{ni} = \int_0^{\frac{2\pi}{\Omega_n}} [-2\zeta_{ni}\Omega_n\dot{\eta}_n]\dot{\eta}_i(t), \quad n,i = 1\ldots 2 \tag{27}$$

where the ζ is defined as the apparent (or equivalent) damping coefficient. From (26), term E_{11} is simplified as follows:

$$\begin{aligned} E_{11} = z_1^2 a_1 & [-1.6254\cos(\phi_1 - \xi_1) \\ & +0.0001\cos(\phi_1 + 3\xi_1) + .0041\sin(\phi_1 + 3\xi_1)]. \end{aligned} \tag{28}$$

The energy production from the first mode damping term ε_{11} is simplified as

$$\begin{aligned} \varepsilon_{11} = [& -.005\,34\cos^2\xi_1 - 1.065\sin\xi_1\cos\xi_1 \\ & +.0025\cos 2\xi_1 + 0.5\sin 2\xi_1 - 6.4826]\zeta_{11}z_1^2. \end{aligned} \tag{29}$$

Equating ε_{11} and E_{11}, the apparent damping coefficient ζ_{11}, after considerable manipulations, takes the following form:

$$\zeta_{11} = a_1\frac{T_1\cos\phi_1 + T_2\sin\phi_1}{T_3} \tag{30}$$

where T_1, T_2, and T_3 are

$$\begin{aligned} T_1 &= -1.63\cos\xi_1 + .004\sin 3\xi_1 + .000\,03\cos 3\xi_1 \\ T_2 &= -1.63\sin\xi_1 + .004\cos 3\xi_1 + .000\,03\sin 3\xi_1 \\ T_3 &= -6.48 - .033\sin 2\xi_1 + .000\,04\cos 2\xi_1. \end{aligned}$$

The other three terms in the damping matrix ζ_{22}, ζ_{12}, ζ_{21} are evaluated in the same fashion using (26) and (27).

In order to dissipate energy, the system's overall damping matrix must be positive–definite. Matrix $A \in R^{n\times n}$ is said to be positive–definite matrix if $x^T Ax > 0$ for all nonzero $x \in R^n$. The Sylvester's criterion is used to check the definiteness of the matrix.

The set of the phase angles ϕ_n and ξ_n that maximize the diagonal elements for unity gains a_1 and a_2

$$\zeta_v = \begin{bmatrix} a_1 f(\phi_1,\xi_1) & a_2 f(\phi_2,\xi_1,\xi_2) \\ a_1 f(\phi_1,\xi_1,\xi_2) & a_2 f(\phi_2,\xi_2) \end{bmatrix} \tag{31}$$

are found to be equal. Fig. 1 shows the variation of the apparent damping matrix diagonal terms with the phase shift angles ϕ_1 and ϕ_2. For the set of the phase shift angles, the values of the

TABLE I
α_n, θ_n NUMERICAL VALUES [1], [3]

Mode	ω_n	α_n	θ_n	$\Omega_n = \omega_n - \theta_n$
First	1	0.005	0.00250	0.9975
Second	2	-0.05	0.00375	1.996

apparent damping coefficients in terms of the controller's gains are summarized in (32)

$$\zeta_v = \begin{bmatrix} 0.2523a_1 & 0.0712a_2 \\ 0.3089a_1 & 0.064\,68a_2 \end{bmatrix}. \tag{32}$$

Having found the angles that maximize the apparent damping factors for unity gains, let us now find the constraints on the values of the amplitude factors a_1 and a_2 required to achieve a stable system. For the first two modes, the numerical values of the damping matrix D are taken from [13], and the system damping coefficients are shown in matrix ζ

$$D = \begin{bmatrix} -.01 & .007 \\ .01 & .1 \end{bmatrix} \Rightarrow \zeta = \begin{bmatrix} -.005 & .0018 \\ .005 & .025 \end{bmatrix}. \tag{33}$$

To drive all the states to the equilibrium, the energy must be continuously dissipated, that is, the controller must supply the necessary damping to offset any energy production from the system. In terms of the matrix algebra, the continuous energy dissipation requires that the overall damping matrix be positive–definite. To achieve this objective, the matrix ζ_{sys} defined in the following must always be positive–definite

$$\zeta_{sys} = \zeta + \zeta_v \tag{34}$$

that is

$$\zeta_{sys} = \begin{bmatrix} -0.005 + 0.2523a_1 & 0.0018 + 0.0712a_2 \\ 0.005 + 0.3089a_1 & 0.025 + 0.064\,68a_2 \end{bmatrix}. \tag{35}$$

Using Sylvester's criterion, the following conditions must be satisfied to guarantee the positive definiteness of the damping matrix, hence, establishing the constraints on the controller gains that guarantee the stability

$$-0.005 + 0.2523a_1 > 0 \Rightarrow a_1 > 0.0195 \tag{36}$$

The second condition is

$$0.025 + 0.064\,68a_2 > 0 \Rightarrow a_2 > -0.4 \Rightarrow a_2 > 0 \tag{37}$$

and finally, the determinant of ζ_{sys}

$$5.8a_1 - 5.7a_1a_2 - 0.68a_2 - 0.13 > 0. \tag{38}$$

When the amplitude and the frequency of the system vary slowly with time, which is the case in our system, the control law can be better implemented in a feedback form [12]. Using the fact that the optimal angles $\phi_n = \xi_n$, (21) can now be rewritten for the optimal energy dissipation as

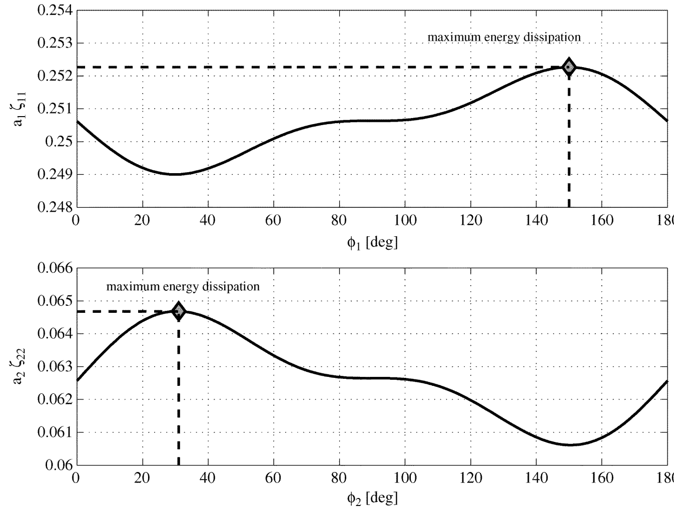

Fig. 1. Apparent damping factor.

$$\begin{aligned} v_n &= a_n \sin(2\Omega_1 t + \phi_n + \xi_n) \\ &= a_n \sin(2\Omega_n t + 2\phi_n) \\ &= 2a_n \sin(\Omega_n t + \phi_n)\cos(\Omega_n t + \phi_n). \end{aligned} \tag{39}$$

Using the time averaged quantities

$$\eta_n(t) = z_n e^{\alpha_n t} \sin\left((\omega_n - \theta_n)t + \xi_n\right) \tag{40}$$
$$\dot{\eta}_n(t) = z_n \omega_n e^{\alpha_n t} \cos\left((\omega_n - \theta_n)t + \xi_n\right) \tag{41}$$

the controller can be written in the following form:

$$\begin{aligned} v_n &= \frac{2a_n \left[\overbrace{z_n e^{\alpha_n t} \sin(\Omega_n t + \phi_n)}^{\eta} \overbrace{z_n e^{\alpha_n t} \omega_n \cos(\Omega_n t + \phi_n)}^{\dot{\eta}} \right]}{\left[z_n^2 e^{2\alpha_n t} \omega_n \underbrace{\left(\sin^2(\Omega_n t + \phi_n) + \cos^2(\Omega_n t + \phi_n)\right)}_{=1} \right]} \\ &= \frac{2a_n \eta_n \dot{\eta}_n}{\omega_n \left[\eta_n^2 + \frac{\dot{\eta}_n^2}{\omega_n^2} \right]} \end{aligned} \tag{42}$$

The system must damp out if the energy production due to the controller is more than that of any other energy source in the system equations. With the controller defined previously, the overall system equations (24) can now be written as

$$\begin{aligned} \ddot{\eta}_n + \omega_n^2 \eta_n = &- \sum_{i=1}^{2} [D_{ni}\dot{\eta}_i + E_{ni}\eta_i] \\ &- \sum_{i=1}^{2}\sum_{j=1}^{2} [A_{nij}\dot{\eta}_i\dot{\eta}_j + B_{nij}\eta_i\eta_j] \end{aligned}$$

$$- 2b_n \left[\frac{a_1}{b_1} \frac{\eta_1^2 \dot{\eta}_1}{\omega_1 \left(\eta_1^2 + \left(\frac{\dot{\eta}_1}{\omega_1} \right)^2 \right)} + \frac{a_2}{b_2} \frac{\eta_2^2 \dot{\eta}_2}{\omega_2 \left(\eta_2^2 + \left(\frac{\dot{\eta}_2}{\omega_2} \right)^2 \right)} \right]. \quad (43)$$

If secondary fuel is used as control input, the following relation can be used to determine the flow rate of the secondary control fuel [13]:

$$u(t) = \frac{\bar{R} \Delta H_c}{\bar{a}^2 \bar{C}_v} \frac{\partial \dot{m}_{\text{in}}}{\partial t} \quad (44)$$

where $\bar{C}_v$ is the constant volume specific heat of the fuel, ΔH_c is the heat of combustion of the fuel, and $\bar{R}$ is the gas constant of the mixture. Therefore, the control input can be expressed as

$$\frac{\partial \dot{m}}{\partial t} = \frac{-2b_n \bar{a}^2 \bar{C}_v}{\bar{R} \Delta H_c} \left(\frac{a_1}{b_1} \frac{\eta_1^2 \dot{\eta}_1}{\omega_1 \left(\eta_1^2 + \left(\frac{\dot{\eta}_1}{\omega_1} \right)^2 \right)} + \frac{a_2}{b_2} \frac{\eta_2^2 \dot{\eta}_2}{\omega_2 \left(\eta_2^2 + \left(\frac{\dot{\eta}_2}{\omega_2} \right)^2 \right)} \right). \quad (45)$$

V. Optimal Controller Gains and Simulations

In any control system design, it is always desired to achieve the control objectives in a timely and an effective manner. Factors such as settling time, overshoot are of prime importance.

The design process usually involves optimizing a certain performance index which is calculated or measured to evaluate the system's performance. Some of the widely used indexes include the integral of square error, ISE. Another performance criterion is the IAE. To reduce the contribution of the large initial error to the value of the performance integral, as well as to emphasize the error occurring later in the response, [14] proposed the integral of the time multiplied by absolute error, $ITAE$. The previous three indexes are defined as

$$\text{ISE} = \int_0^{t_f} e^2(t) dt \quad (46)$$

$$\text{IAE} = \int_0^{t_f} |e(t)| \, dt \quad (47)$$

$$\text{ITAE} = \int_0^{t_f} t \, |e(t)| \, dt \quad (48)$$

where the upper limit t_f is a finite time chosen such that the integral approaches steady-state value, it is usually chosen as the settling time. The $ITAE$ performance provides the best selectivity of the performance indices; that is the minimum value of the integral is readily discernible as the system parameters are

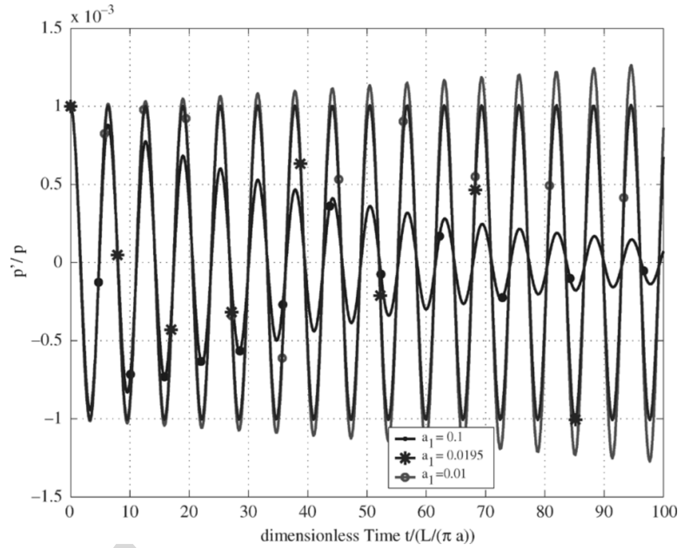

Fig. 2. Controller gains validation $a_2 = 0$.

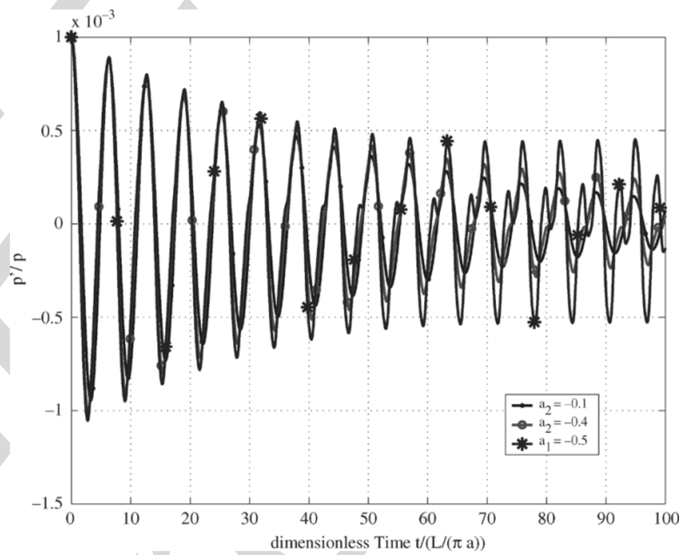

Fig. 3. First mode evolution for different a_2 gains, and $a_1 = 0.1$.

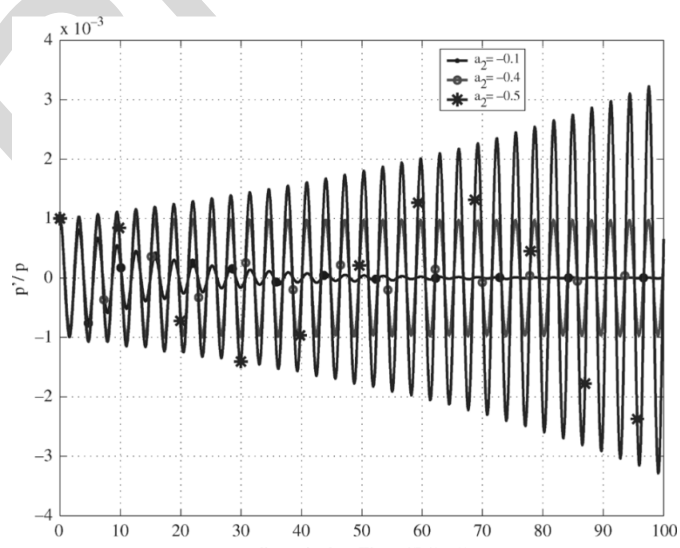

Fig. 4. Second mode evolution for different a_2 gains, and $a_1 = 0.1$.

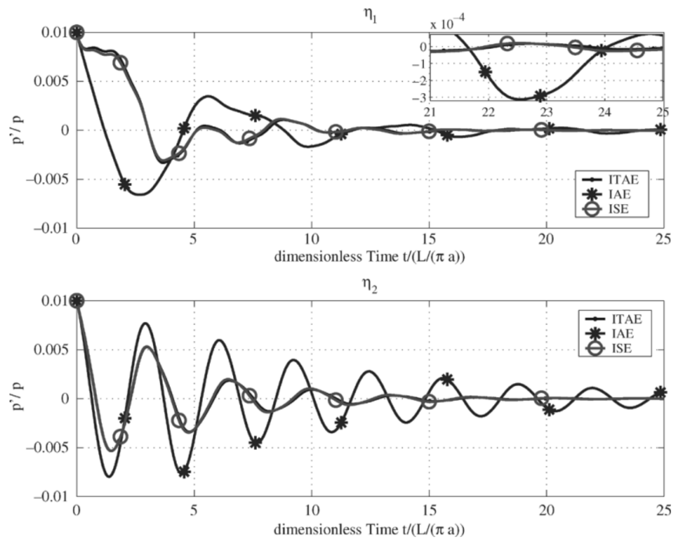

Fig. 5. Closed-loop performance using optimal controller gains.

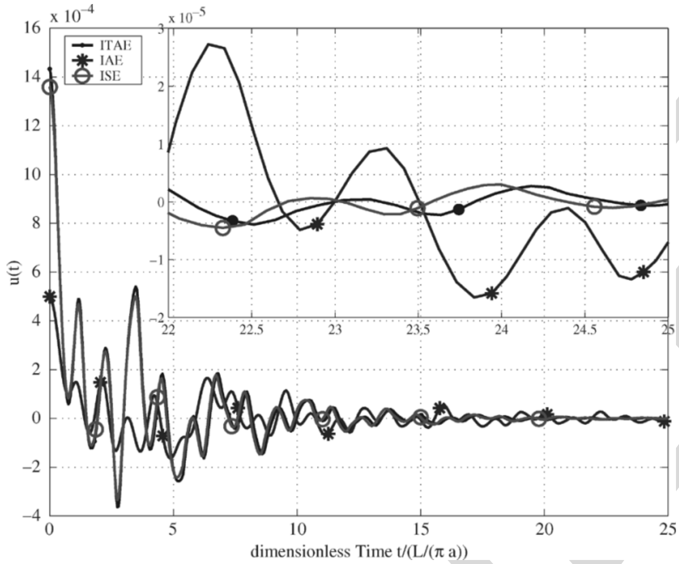

Fig. 6. Control evolution for different performance criterion.

varied [15]. Clearly, one can obtain numerous indexes based on the various combinations of the system variables and time.

A. Controller Gains Constraints Validation

In this section, the constraints on the controller gains given in (36)–(38) are validated, then these gains are optimized using the indexes mentioned previously. Throughout the simulations, it is assumed that the control action is provided by a single point actuator and the conditions in the combustor are monitored by one collocated sensor [16].

To validate the constraints on the controller gain factors according to Sylvester's criterion given in (36)– (38), a set of controller gains combinations are studied.

Following are some of the critical combinations of the controller gains:

1) Variable a_2 Set to Zero: The condition set on the value of the gain a_1 by (36)–(38) requires this factor to be greater than 0.0195 to ensure the system stability. To confirm this condition, three values of this gain are considered, namely, 0.1, 0.0195, 0.01.

As expected from the conditions, the controller damps out the undesired oscillations for values $a_1 > 0.0195$. The rate of attenuation of the oscillations is directly proportional to the value of the gain. For a value of $a_1 = .0195$ the controller forces the system to behave like an undamped second-order system for the given simulation time span. For the third case, $a_1 < .0195$, the controller dose not suppress the oscillation meaning more energy is added to that mode. Fig. 2 shows the time trace of the first mode for the three gain values discussed previously.

2) Variable a_1 Set to 0.1: Using the same approach discussed previously, the system must be stable for gain values ($a_2 > -0.40$) and undamped or unstable otherwise, provided that gain a_1 satisfies the stability conditions. Similar results to those of the gain factor a_1 are obtained in this case, where again, three different values of a_2 are considered. Figs. 3 and 4 show the time trace of the first and the second mode amplitudes for values of $a_2 = -0.1, -0.4, 0.5$ and $a_1 = 0.1$. The system response is satisfactory for the value of $a_2 = -0.1$, and both modes damp out.

For the value of $a_2 = -0.40$, similar to the limiting value of a_1, the second mode undergoes a very slow change in amplitude as it is clear in Fig. 4. The first mode seems to damp out in this case, but longer simulation shows that it starts to go unstable. Finally, setting $a_2 < -0.4$, results in exciting the second mode. The pumped energy in the mode supersedes the dissipated counter part. The first mode damps out at the beginning of the simulation, but it gets excited later.

B. Optimal Controller Gains

Having found the parametric control law and the constraints on the controller parameters a_1 and a_2, the next logical step is to find an optimal set of these parameters that minimizes a given performance index. As mentioned at the beginning of this section, ITAE, IAE, and ISE indexes are used as a performance measure. Since, only two variable had to be solved for to minimize the ITAE, IAE, and the ISE cost functions, an array of combinations of the gains of the controllers which satisfy the stability constraints are used to generate the cost surface from which the optimal controller gains are determined. The cost surfaces are smooth and a gradient based optimization approach can also be used to arrive at the controller gains. The optimal values of the controller parameters using these indexes are summarized in Table II.

To study the effects of the controller parameters on each mode, two scenarios are considered:

In the first case, it is assumed that the lower two modes, where the first is unstable and the second is stable, are controlled and the higher modes are considered to be residual. The optimal controller parameter values for this case are shown in the second column of Table II. Figs. 5 and 6, respectively, illustrate the time trace of the controlled modes and the associated control evolution. In terms of the settling time, the performance of the ITAE and ISE based controllers is better than the IAE based controller as it is clearly depicted in these figures.

The second scenario considers the case in which the unstable first mode is controlled and the other three modes are considered

TABLE II
OPTIMAL CONTROLLER GAINS AND COST FUNCTION

Criterion	Optimal gains a_1 and a_2			Optimal a_1 while $a_2 = 0$	
	a_1	a_2	Cost Function	a_1	Cost Function
$ITAE$	1.7195	2.4500	1.1937×10^{-2}	1.1345	9.2960×10^{-2}
IAE	0.9595	0.5200	2.4808×10^{-6}	1.1005	2.8522×10^{-6}
ISE	1.6995	2.2600	3.9019×10^{-3}	1.1195	6.9194×10^{-3}

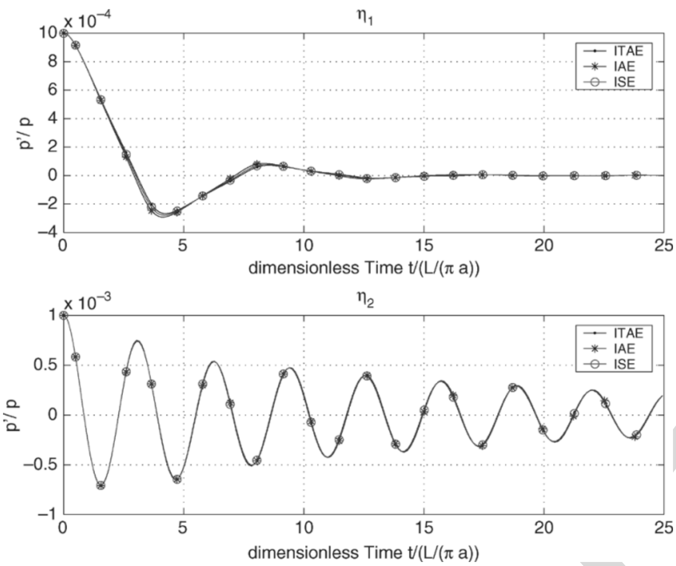

Fig. 7. Closed-loop response using optimal gains a_1 only.

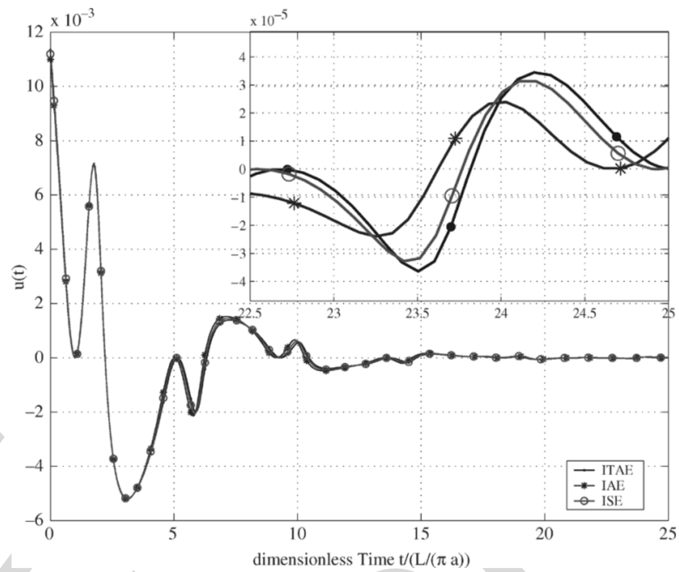

Fig. 8. Control evolution considering the first mode only.

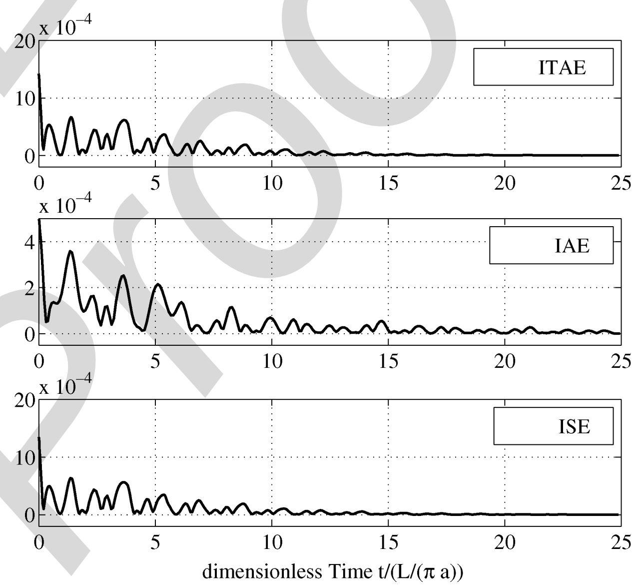

Fig. 9. Secondary control fuel mass flow rate.

to be residual. The optimal values of the controller parameter a_1 are shown in the third column of Table II. The time trace of the first two modes and the control evolution using this parameter are shown in Figs. 7 and 8, respectively. It is evident from both the optimal parameter values and the simulation results that there is no significant difference in the first mode and the control evolution for the three indices, however, comparing the second mode time trace with the first case in which both modes are controlled shows that this mode is not significantly influenced by controlling the unstable first mode despite the fact that they are coupled. This indicates the weak coupling between these modes. It is also clear that the difference in secondary (control) fuel flow between the three indexes is negligible. For the case where the two modes are controlled, the amount of the consumed control fuel given in (45) is found by evaluating the area under the ITAE, IAE, and ISE control evolution curves shown in Fig. 9. The amount of the secondary fuel for the ITAE, IAE, and ISE performance measures are found to be 0.019, 0.0012, and 0.0020, respectively. These values demonstrate the similarity between the performance of ITAE and ISE indices, further they exhibit the trade off between the amount of the control fuel and the residual vibration as it is clearly illustrated in Fig. 5.

VI. CONCLUSION

The model-based parametric controller presented in this work is found to be successful in stabilizing the system. It is based on the selection of phase shift that maximizes the energy dissipation and a optimal gain factors that minimizes some prespecified performance index. Then, based on the desired performance of the control system, the system's parameters may be adjusted to provide the required response. If optimal phase assumes certain mathematical relation, the physical implication of the optimal phase is that heat addition produced by the feedback law should be out of phase with respect to the acoustic oscillation in conjunction with anti sound principles proposed by the famous Rayleigh Criterion. In practice, however, the implementation of the optimal phase is challenging. In real life problems, however, a time delay associated with the modulated control fuel ignition exists. It is very difficult to estimate this delay because it is a function of the equivalence ratio and it is very sensitive to the flow disturbances. Of course, we should remember actuators and sensors limitations in terms of their delays and bandwidths. For these reasons, the control phase could be tuned online for different operation conditions.

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