# Stability Analysis of Nonlinear Parametric and Coulomb Control Systems

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*Abstract:* Stability analysis of nonlinear control systems using Lypunov's direct method is reported in this paper. A general nonlinear controlled system and two different types of controllers are examined for stability analysis. The first category is the parametric controller. Importance of parametric controllers is illustrated through examples of various engineering systems. A general control algorithm is derived, and stability of the closed-loop system is examined. Stability under small external disturbances is also examined. The second category of controllers examined in this paper is the Coulomb-type controllers. Stability analysis is performed using Lypunov's direct method. Stability under a general forcing excitation is examined for linear systems with Coulomb controllers.

*Key Words:* Lypunov's method, parametric systems, coulomb dampers, nonlinear control

## 1. INTRODUCTION

Nonlinear systems are frequently encountered in dynamic modeling and control of engineering systems. It is always desirable to establish stability of the controlled system to assess the performance of the controller over a range of system parameters. Stability analysis of engineering systems has been studied in great detail in the literature. Nayfeh and Mook (1979) discuss stability analysis of various types of systems, linear and nonlinear, using a number of different approaches. Slotine and Li (1991) discuss stability analysis using Lypunov's method for several different linear and nonlinear systems. A generalized theory for studying stability of nonlinear systems is not available because of the complexity of nonlinear equations; each nonlinear system has to be treated differently for stability analysis. This paper discusses stability of a general nonlinear system, using Lypunov's direct method, for two different types of controllers encountered frequently in engineering systems.

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## 2. GENERAL NONLINEAR SYSTEM

Consider a general second-order nonlinear system

$$\ddot{x} + f(x,\dot{x}) + g(x) + h(x,\dot{x}) = F(t), \tag{1}$$

where $f(x,\dot{x})$ is the nonlinear damping term, $g(x)$ is nonlinear stiffness, $F(t)$ is the external disturbance or direct forcing function, and $h(x,\dot{x})$ is a general nonlinear control term. Assume that

$$\begin{aligned} \dot{x}f(x,\dot{x}) &\geq 0; \quad \dot{x} \neq 0 \\ xg(x) &> 0; \quad x \neq 0. \end{aligned} \tag{2}$$

These sign conditions imply that $f(x,\dot{x})$ and $g(x)$ represent damping and spring effects, which are usually true for physical systems. Also, the preceding assumption implies that stiffness-damping pair $f(x,\dot{x})$ and $g(x)$ forms a stable system if the control term $h(x,\dot{x})$ is absent.

This paper presents the use of Lypunov's direct method in establishing the stability of equation 1 for two different types of controllers. The first category discussed in this paper is the parametric controller where $h(x,\dot{x})$ is defined as

$$h(x,\dot{x}) = v(t)x. \tag{3}$$

Here $v(t)$ is the parametric feedback of the controller. The second category is coulomb-type controllers, where

$$h(x,\dot{x}) = Csgn(\dot{x}), \tag{4}$$

*sgn* being the standard signum function.

## 3. PARAMETRIC CONTROLLERS

Parametric controllers are descriptive of cases where control application appears as a time varying modification of a system parameter. The phenomenon of parametric vibrations occurs frequently in various fields of science and engineering and is dealt with in various textbooks, for example, Nayfeh and Mook (1979), Cartmell (1990), and Minorsky (1962). This phenomenon was first observed by Faraday in 1831 during his studies on surface waves in a fluid-filled cylinder. Melde in 1859 demonstrated this effect by a more readily observable example: a string tied between a rigid support and a tuning fork. Rayleigh in 1887 was perhaps the first person to provide a theoretical basis for these effects. Ever since then, parametric vibrations have been studied in connection with various branches of physics and engineering. The problem of parametric resonance and stability has been studied in connection with surface waves, string vibrations, Euler columns, electric circuits, vortex shedding vibrations, and so on. Notable contributions have been made by Faraday, Rayleigh, Raman, Poincaré, and Timoshenko to the enrichment of this area.

The most distinguishing feature of parametric vibrations is that if a parameter (such as stiffness or inertia) of an oscillatory system is made to vary at $2f$, $f$ being the frequency of the system, then the system begins to oscillate at a frequency of $f$ and resonance is established. In general, parametric resonance can occur when the frequency of the oscillatory parameter is an integer multiple of $f$. The resonant vibrations of a parametric system are unstable and grow until other nonlinearities of the system come into play and saturate the system. Some of the generic properties of parametric vibrations are noted below:

- Bandwidth region of dynamic instability,
- May occur in direction normal to the excitation,
- May occur at frequencies other than natural frequencies,
- Amplitude of vibration increases exponentially under resonance (as against linear increase in ordinary resonance),
- Bistable system, and
- Parametric excitation is not effective until the equilibrium position is perturbed, after which oscillations build up. This is in contrast to forced vibrations where oscillations can be started at the equilibrium position.

Mathematically, parametric excitations occur as time-varying coefficients in a differential equation. The dynamic equation of the system remains homogeneous as against the inhomogeneous ones due to forced excitations. Therefore, parametric vibrations are mathematically modeled with equations that are homogeneous and have time-dependent parameters. The equation itself can be linear or nonlinear depending on other features of the system. Mathematically, linear parametric differential equations are called Hill's equation or Matthiu's equation. Nonlinearities occur in most real systems and can modify response of parametric systems significantly.

A typical linear parametric system can be written as

$$\ddot{x} + 2\xi\omega\dot{x} + [\omega^2 + \nu(t)]x = 0. \tag{5}$$

This is the equation of a damped linear oscillator forced parametrically due to the term $\nu(t)x$, a time-dependent stiffness term. Linear parametric equations usually occur as a simplification of nonlinear ones (e.g. beam with a tip mass). The stability characteristics of a linear parametric equation can be determined by Floquet theory (Nayfeh and Mook, 1979). In general, resonance of linear parametric system occurs when frequency of $\nu(t)$ is a multiple of $\omega$. The resonant vibrations are unstable for an undamped system, and for systems with viscous damping the vibrations become unstable when a threshold amplitude (a function of damping) is exceeded.

A classic example of nonlinear parametric problems is a pendulum with a moving support. The equation of motion for this system is

$$\ddot{\theta} + 2\xi\omega\dot{\theta} + [\omega^2 + \nu(t)]\sin\theta = 0, \tag{6}$$

where $\nu(t)$ is the vertical support motion. This problem has been studied extensively over the years and has several applications in mechanical engineering and control systems. The preceding equation can be converted to linear or different nonlinear forms, depending on assumptions made for the $\sin\theta$ term. The nonlinearity of this system ensures bounded solution (linearized solution is unbounded at resonance). Related problems of compound and gyroscopic pendulums also fall into the same category. Here, the support motion can be used as a controller if designed suitably.

Vibrations of a string with a moving support is also a classic parametric vibration problem. This system exhibits cubic stiffness type nonlinearity, and the equation of motion can be written as

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x + \epsilon x^3 + \nu(t)x = 0, \tag{7}$$

where $\nu(t)$ is the support motion. The resonant vibrations occur in the direction normal to the direction of support motion. The nonlinearity saturates the otherwise unstable parametric vibrations. Fujino, Warnitchai, and Pacheo (1993) have shown that a properly chosen $\nu(t)$ can be used to control the string vibrations. They have shown that energy of system is dissipated when $\nu(t)$ is at twice the natural frequency and is out of phase with string oscillations.

A more recent application of parametric vibrations is in vortex shedding vibrations of cylinders. Billah (1989) proposed the equation

$$\ddot{x} + 2\xi\omega\dot{x} + \epsilon x^2\dot{x} + \omega^2 x + u(t)x = 0 \tag{8}$$

as a model for vortex-induced vibrations. Again, as in the string problem, the vibrations are normal to the direction of excitations. This equation has nonlinear damping with a time-dependent variation in stiffness. The vibrations tend to a limit cycle during resonance due to nonlinear amplitude-dependent dissipation of energy.

It is seen through above examples that parametric excitations in linear as well as nonlinear systems occur frequently. Many of these systems have unstable or large resonant vibrations depending on the nonlinearity of the system. Hence, control of such systems is of engineering importance. The control problem for these systems presents an exciting possibility of using the parametric excitation itself for useful purposes. If support motion is controlled in a proper manner, then vibrations of a pendulum or a string can be stabilized. In such cases, positive effect of parametric vibrations in dissipating system energy is utilized. It is well established that if parametric excitation, at twice the natural frequency of the system, is out of phase with motion, then the energy is dissipated whereas for in-phase motions, the energy is added to the system. The control problem is thus to ensure that parametric excitation is out of phase with the system oscillations, ensuring dissipation of system energy at all times.

Based on the foregoing discussion, a general class of parametric system can be represented by

$$\ddot{x} + f(x, \dot{x}) + g(x) + \nu(t)x = 0, \tag{9}$$

which is a modified form of equation (1) with $F(t) = 0$ and with $\nu(t)x$ representing the parametric excitation. It should be noted that this is a fairly general equation and covers all the cases discussed earlier. The higher-order parametric terms such as $\nu(t)x^3$ are not included. Thus, this equation describes vibrations of a general class of nonlinear systems subjected to linear parametric excitation. If the term $\nu(t)$ can be altered (like changing the support motion) in a real system, then it becomes the control term of the equation. The proper selection of $\nu(t)$ can ensure system stability and reduction in response.

### *3.1. Control Algorithm for* $v(t)$

A number of control methods have been proposed for nonlinear systems, for example, Mohler (1991) and Slotine and Li (1991). However, control methods for nonlinear systems are not general and each method can be applied to a specific class of nonlinear systems only. Relatively few specific works are available on control of parametric systems. Some discussion on the control of such systems is available in Mohler (1973). In this study, a control approach suggested by Fujino, Warnitchai, and Pacheo (1993) will be adopted, which provides a controller that uses dissipative effects of parametric systems. The authors developed this controller in conjunction with cable vibrations where equations are nonlinear with cubic stiffness (a duffing oscillator). A derivation of this controller is presented here.

In equation (9), $v(t)$ appears as a parametric stiffness term. As noted earlier, $v(t)$ can be used to dissipate the energy of the system if its phase and frequency are properly tuned (Nayfeh and Mook, 1979). For simplicity, assume that the controlled vibrations are harmonic with frequency $\omega$. Also, assume that actuator displacement $v(t)$ varies sinusoidally, then

$$x = a_x \cos(\omega t) \tag{10}$$

$$v = a_v \cos(n\omega t + \phi), \tag{11}$$

where $a_x$ and $a_v$ are amplitudes of $x$ and $v$, respectively, and $n$ is an integer. It is well known that parametric resonance occurs most readily when the frequency of parametric excitation is twice that of the natural frequency, hence $n = 2$ is a logical choice in equation 11.

Now let us consider per cycle energy production of the term $v(t)x$ in equation (9). Let $E$ be the energy production per cycle, then

$$E = \int_0^{2\pi/\omega} (-v(t)x)\dot{x}dt \tag{12}$$

$$\text{or, } E = \int_0^{2\pi/\omega} [-a_v \cos(2\omega t + \phi)a_x \cos(\omega t)](-a_x \omega \sin(\omega t))dt$$

$$\text{or, } E = a_v a_x^2 \omega \int_0^{2\pi/\omega} \cos(2\omega t + \phi) \cos(\omega t) \sin(\omega t)dt$$

$$\text{or, } E = -\frac{1}{2}\pi\, a_v a_x^2 \sin\phi. \tag{13}$$

Thus energy production due to $v(t)$ is negative (dissipation) when $\sin\phi > 0$ and is positive when $\sin\phi < 0$. Similarly, energy production due to viscous damping would be

$$E = \int_0^{2\pi/\omega} (-2\xi\omega\dot{x})\dot{x}dt \tag{14}$$

$$
\begin{aligned}
\text{or,}\quad E &= \int_0^{2\pi/\omega} (-2\xi\omega)(a_x\omega\sin(\omega t))^2 dt \\
\text{or,}\quad E &= -2\xi\omega^3 a_x^2 \int_0^{2\pi/\omega} \sin^2(\omega t)dt \\
\text{or,}\quad E &= -2\pi\xi\omega^2 a_x^2. \qquad (15)
\end{aligned}
$$

Equating equations (15) and (13) yields

$$\xi_v = \frac{1}{4\omega^2} a_v \sin\phi, \qquad (16)$$

where $\xi_v$ is the apparent damping coefficient due to $v(t)$. Now for control purposes, we want $\xi_v > 0$ hence $\sin\phi > 0$. Figure 1 shows variation of additional damping with $\phi$ for unit $\omega$ and $a_v$. The damping coefficient is optimal when $\sin\phi = 1$, that is, $\phi = \pi/2$. Thus we can rewrite $v(t)$ for optimal energy dissipation as

$$v(t) = a_v \cos(2\omega t + \pi/2). \qquad (17)$$

When amplitude and frequency of $x$ varies slowly with time, the above equation can be better implemented in a feedback form as

$$
\begin{aligned}
v(t) &= a_v\cos(2\omega t + \pi/2) = -a_v\sin(2\omega t) \\
\text{or,}\quad v(t) &= -2a_v\sin(\omega t)\cos(\omega t) \\
\text{or,}\quad v(t) &= \frac{2a_v}{\omega}\frac{(-a_x\sin(\omega t))(a_x\cos(\omega t))}{a_x^2} \\
\text{or,}\quad v(t) &= \frac{2a_v}{\omega}\frac{x\dot{x}}{[a_x^2\cos^2(\omega t) + a_x^2\sin^2(\omega t)]} \\
\text{or,}\quad v(t) &= \frac{2a_v}{\omega}\frac{x\dot{x}}{[x^2 + \frac{\dot{x}^2}{\omega^2}]}. \qquad (18)
\end{aligned}
$$

Thus, the algorithm in equation (18) produces a net damping effect on vibrations of systems modeled by equation (9).

The complete system of equations for a general parametric system can be summarized as

$$\ddot{x} + f(x,\dot{x}) + g(x) + v(t)x = 0, \qquad (19)$$

with the controller as

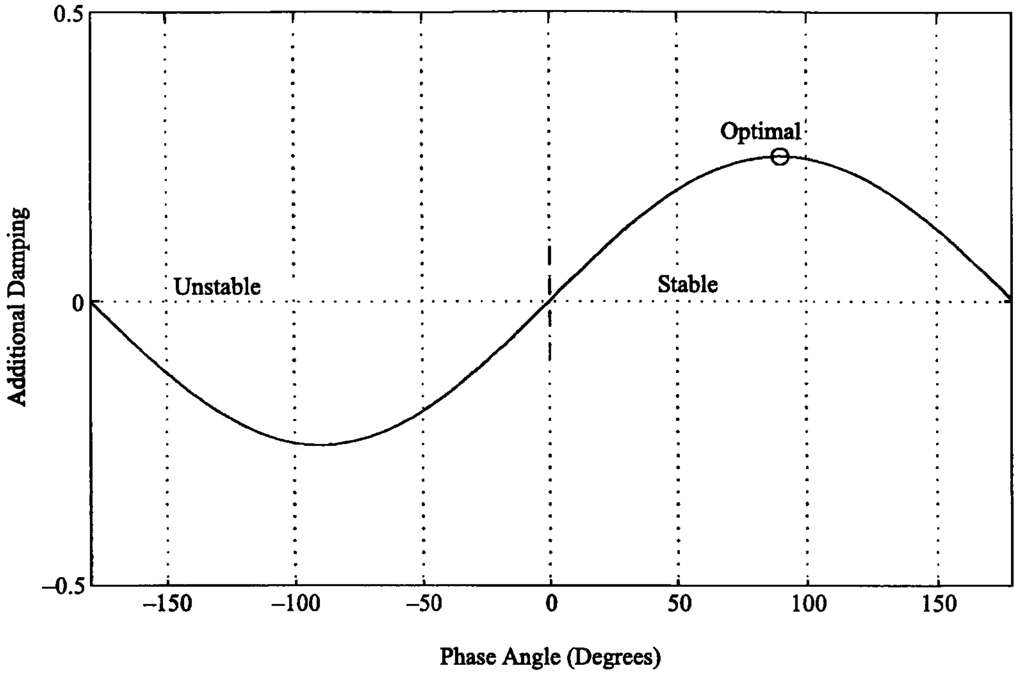


Figure 1. Variation of $\xi_v$ with $\phi$.

$$v(t) = \frac{2a_v}{\omega} \frac{x\dot{x}}{\left[x^2 + \frac{\dot{x}^2}{\omega^2}\right]}, \tag{20}$$

where $a_v$ is the controller gain. It is noted that expression in equation (20) is not singular, since when $(x, \dot{x}) \rightarrow (0, 0)$ then $v(t) \rightarrow 0$ in the limit. It is evident that derivation of the controller is independent of the form of $f(x, \dot{x})$ or $g(x)$, but the stability of the system depends on these terms. If the net energy dissipation due to $v(t)$ is more than the production due to other terms, then the system will damp out.

### *3.2. Stability Analysis*

As mentioned earlier, the stability of equation (19) with equation (20) as the controller depends on the nature of nonlinear damping and stiffness in the system. The conditions required for asymptotic stability of this system will now be investigated using Lypunov's direct method. The system described by equation (19) under a general case of parametric vibrations is nonautonomous and hence it might be difficult to find a Lypunov's function for the system. For this purpose, stability of the closed-loop dynamics of the system will be examined, that is,

$$\ddot{x} + f(x, \dot{x}) + g(x) + \frac{2a_v}{\omega} \frac{x^2\dot{x}}{\left[x^2 + \frac{\dot{x}^2}{\omega^2}\right]} = 0. \tag{21}$$

The advantage of using equation (21) is that the time does not appear explicitly in the equation and thus is an autonomous equation. The closed-loop system of this form can be treated by Lypunov's direct method (Slotine and Li, 1991). Lypunov's direct method is an extension of a fundamental physical observation: if the total energy of the system is continuously dissipated, then the system must eventually settle down to an equilibrium point.

The equilibrium point for equation (21) can be easily determined by inspection to be $(x, \dot{x}) = (0, 0)$. Consider a positive definite function for this system,

$$V = \frac{\dot{x}^2}{2} + \int_0^x g(z)dz, \tag{22}$$

as a choice for the Lypunov function. The time derivative of this function is

$$\dot{V} = \dot{x}\ddot{x} + g(x)\dot{x} \tag{23}$$

$$\text{or,} \quad \dot{V} = \dot{x}[-f(x,\dot{x}) - g(x) - \frac{2a_v}{\omega}\frac{x^2\dot{x}}{[x^2 + \frac{\dot{x}^2}{\omega^2}]}] + g(x)\dot{x}$$

$$\text{or,} \quad \dot{V} = [-f(x,\dot{x})\dot{x} - \frac{2a_v}{\omega}\frac{x^2\dot{x}^2}{[x^2 + \frac{\dot{x}^2}{\omega^2}]}] \leq 0. \tag{24}$$

Thus the function $\dot{V}$ is negative semi-definite since $\dot{V} = 0$ for all $\dot{x} = 0$ and $x \neq 0$. Using invariant set theorem (Slotine and Li, 1991), it can be shown that the largest invariant set in the set formed by points on $\dot{x} = 0$ is the equilibrium point. This means that the state trajectories cannot converge to any point other than the equilibrium point, thus ensuring asymptotic stability. Now, in terms of state vector $\underline{x} = [x; \dot{x}]$, if integral $\int_0^x g(z)dz \to \infty$ as $\| \underline{x} \| \to \infty$, then $V \to \infty$ as $\| \underline{x} \| \to \infty$ and thus $V(\underline{x})$ would be radially unbounded. Summarizing, we have

| | | | |
|---|---|---|---|
| 1) | $V(\underline{x})$ | is positive definite | |
| 2) | Eq. Point | is asymptotically stable | (25) |
| 3) | $V(\underline{x})$ | is radially unbounded. | |

Thus, the equilibrium point is *Globally Asymptotically Stable* based on Lypunov's direct method and the invariant set theorem. This implies that vibrations of this general closed-loop parametric system will always settle down to the equilibrium point, which is the rest position. Hence, the controller in equation (20) produces a globally asymptotically stable closed-loop system for a general class of $f(x, \dot{x})$ and $g(x)$ functions in equation (19).

The above analysis using Lypunov's direct method establishes only the stability of the equilibrium point. The equation treated in this analysis is homogeneous and does not contain any explicit forcing term. In practical applications, the stability of the system due to external forcing perturbations is also desired. The forcing perturbations may occur in engineering systems due to explicit forces or unmodeled dynamics. It is thus desirable to study the stability of equilibrium point under small persistent disturbances. Theorem of Total Stability (Hahn, 1963) can be cited for this purpose, which states that if the equilibrium point (of an autonomous or nonautonomous equation) is uniformly asymptotically stable, then it is

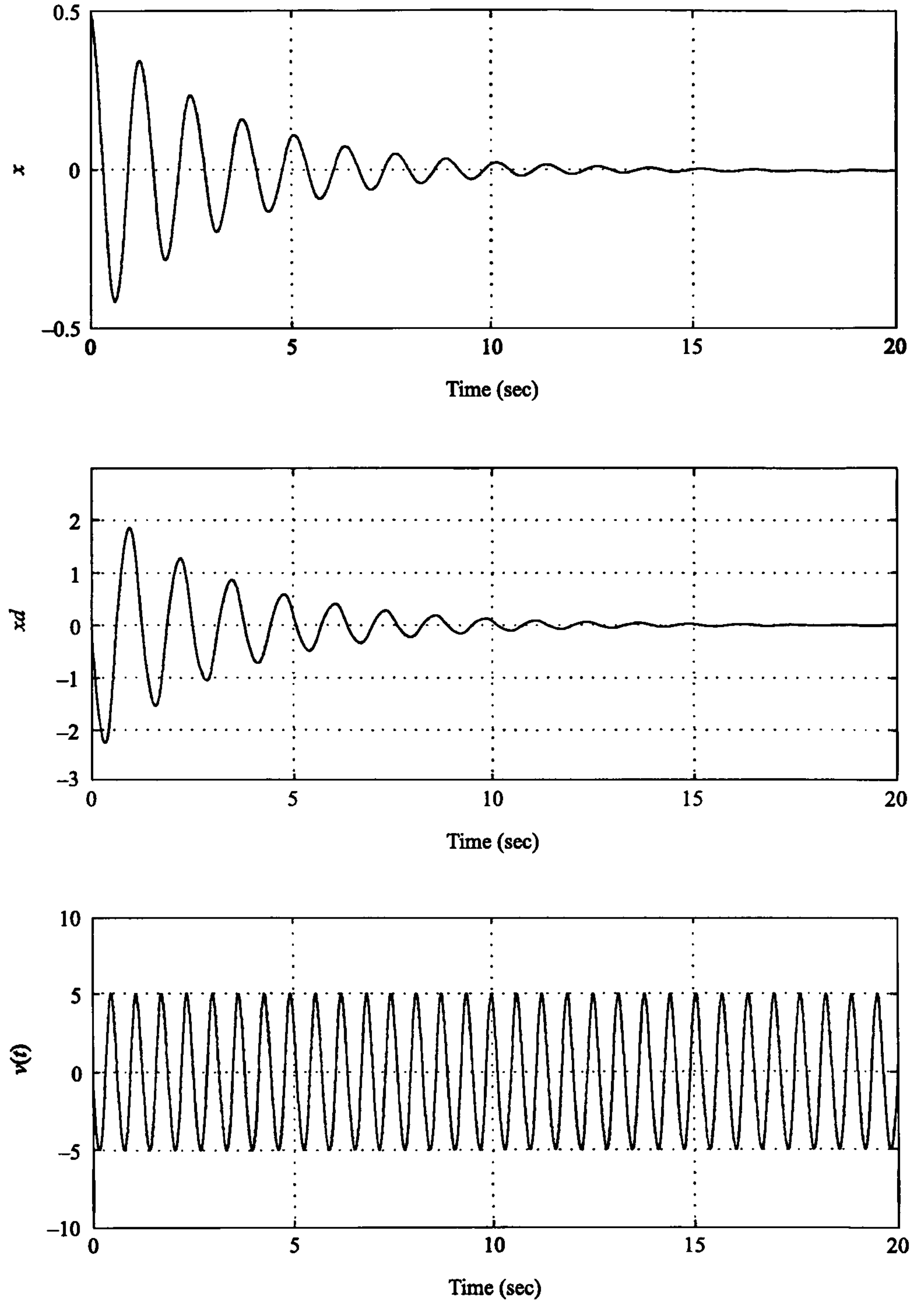


Figure 2. Numerical simulation of a linear system with a parametric controller. $(xd = \dot{x})$

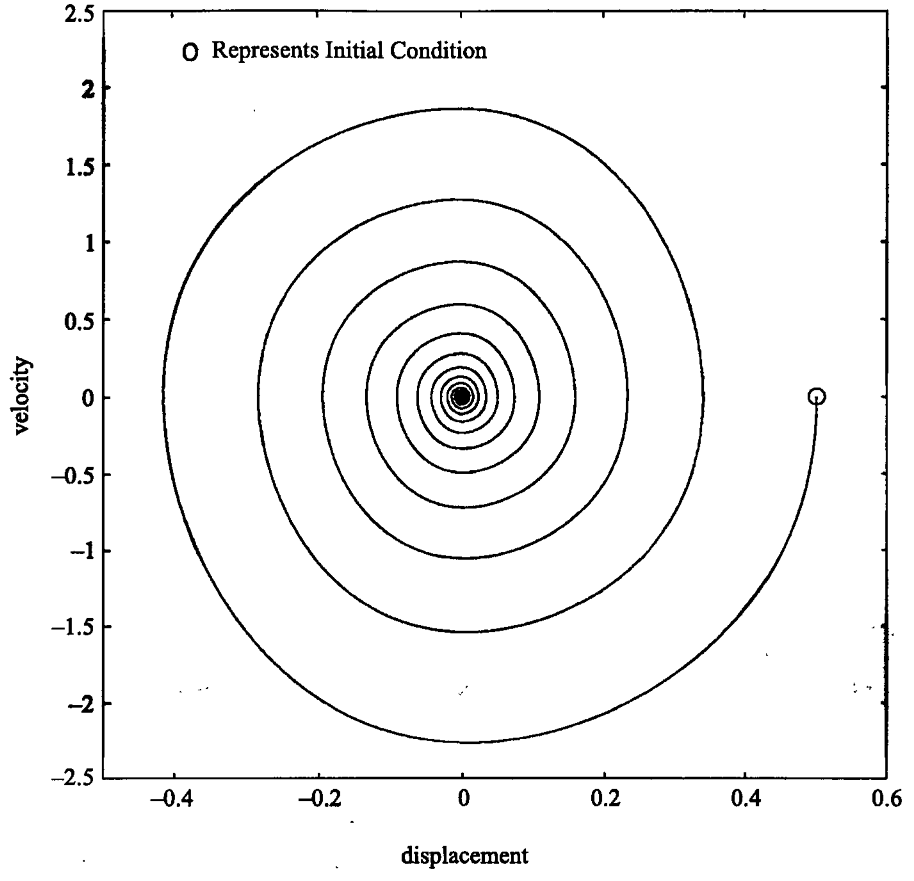


Figure 3. State space of a linear system with a parametric controller.

also totally stable. Here, total stability refers to stability under constantly acting small and bounded perturbations. The system in equation (21) is globally asymptotically stable and, by virtue of being autonomous, it is uniformly globally asymptotically stable. Thus, the preceding theorem guarantees stability of the system under persistent, small, and bounded external disturbances.

### *3.3. Control Examples*

A linear damped system would have $f(x, \dot{x}) = 2\xi\omega\dot{x}$ and $g(x) = \omega^2 x$ in equation (19). Behavior of linear parametric systems can be quantified through Floquet theory (Nayfeh and Mook, 1979). Hagedorn (1981) has shown that vibrations of a linear parametric system are unstable at frequency ratio of 2. The instability diagrams show that for a damped oscillator a threshold exists beyond which the vibrations become unstable. Under a general case of parametric excitation, the response grows unbounded (e.g. $u = 5\cos\omega t$) for this system. The control of such systems is thus of interest. It is straightforward to verify that conditions

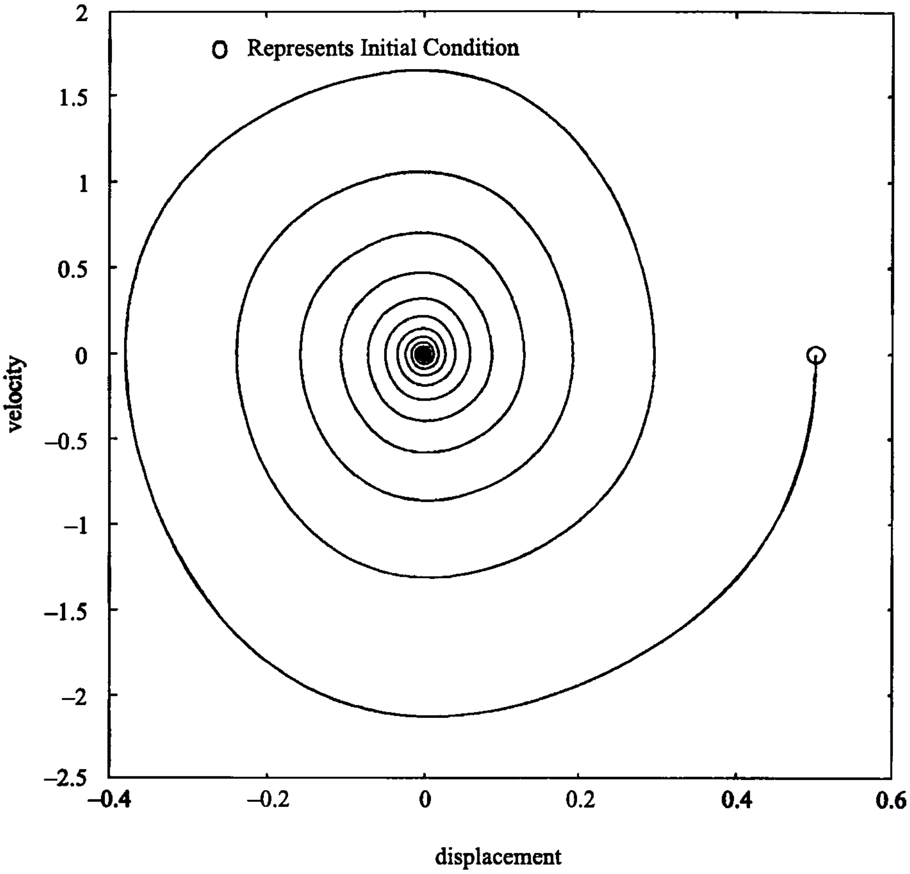


Figure 4. State space of a Vander Pol system with a parametric controller.

in equation (25) are satisfied for this system. Hence, the control law given by equation (20) can be used to stabilize vibrations in a linear parametric system. Results from numerical simulation are presented in Figure 2. This simulation is based on $\xi = 1\%$, $\omega = 5$ rad/s and $a_u = 5$. Figure 3 shows system trajectory in state space for this case.

A vander Pol damping system can be described by $f(x, \dot{x}) = 2\xi\omega\dot{x} + \epsilon x^2\dot{x}$ and $g(x) = \omega^2 x$. The Vander Pol system is relevant in the studies of vortex shedding vibrations. Some properties of this system are presented in Billah (1989). In general, the motion of this oscillator is bounded due to the presence of nonlinear damping. However, a complex stability behavior is exhibited in the parameter space. The conditions of equation (25) can be easily verified. Thus, the controller described by equation (20) is expected to produce stable vibrations. The behavior in state space for a system with $\epsilon = 5.5$, $\xi = 1\%$, $\omega = 5$ rad/s and $a_u = 5$ is presented in Figure 4.

## 4. COULOMB-TYPE CONTROLLERS

The general equation for a Coulomb-type controller is

$$\ddot{x} + f(x,\dot{x}) + g(x) + Csgn(\dot{x}) = F(t), \tag{26}$$

$C$ being a positive constant and $F(t)$ being the external forcing function.

Coulomb-type controllers are also encountered frequently in engineering systems modeled with Coulomb friction. The most common example of such problems is the response of sliding blocks. The problem of a sliding block has been investigated in great detail in connection with civil engineering structures. Westermo and Udwadia (1983) have studied the response of a sliding block under harmonic excitations. Constantinou and Tadjbakhsh (1984) studied response of a sliding structure to filtered random excitations. Jones and Shenton (1990) considered generalized slide-rock response of rigid blocks during earthquakes. Lin et al. (1994) considered sliding motion of anchored rigid blocks modeled with nonlinear equations under random base excitations. Several problems in base isolation are also modeled with Coulomb damping terms (Crandall, Lee, and Williams, 1974).

In sliding mode control problems, a Coulomb-type controller is often used as a sliding surface (Slotine and Li, 1991), giving rise to systems that can be represented by equation (26). In a general case of sliding mode control, equation (26) can be linear or nonlinear depending on other properties of the system.

A Coulomb-type controller is also encountered in control of along-wind motion with active appendages (Soong and Skinner, 1981). The switching function defined as

$$\begin{aligned} S &= 1;\ \dot{x} \leq 0 \\ S &= 0;\ \dot{x} > 0 \end{aligned} \tag{27}$$

is often used for appendage operation. This switching function can be represented in an alternate form as

$$S = \frac{1}{2}[1 - sgn(\dot{x})], \tag{28}$$

which gives rise to an equation with Coulomb-type controller for along-wind motion.

### *4.1. Stability Analysis*

First consider a homogeneous form of equation (26), that is,

$$\ddot{x} + f(x,\dot{x}) + g(x) + Csgn(\dot{x}) = 0. \tag{29}$$

This equation will be used to examine stability of the equilibrium point using Lypunov's direct method.

The equilibrium point for equation (29) can be determined by inspection to be $(x,\dot{x}) = (0,0)$. Consider a positive definite function for this equation,

$$V = \frac{\dot{x}^2}{2} + \int_0^x g(z)dz, \tag{30}$$

as a choice for the Lypunov function. The time derivative of this function is

$$\dot{V} = \dot{x}\ddot{x} + g(x)\dot{x} \tag{31}$$

$$\text{or,} \quad \dot{V} = \dot{x}[-f(x,\dot{x}) - g(x) - Csgn(\dot{x})] + g(x)\dot{x}$$

$$\text{or,} \quad \dot{V} = [-f(x,\dot{x})\dot{x} - C\dot{x}sgn(\dot{x})] \leq 0. \tag{32}$$

Thus, the function $\dot{V}$ is negative semi-definite since $\dot{V} = 0$ for all $\dot{x} = 0$ and $x \neq 0$. Using invariant set theorem (Slotine and Li, 1991), it can be shown that the largest invariant set in the set formed by points on $\dot{x} = 0$ is the equilibrium point. This means that the state trajectories cannot converge to any point other than the equilibrium point, ensuring asymptotic stability. Since $\omega^2 x^2 \rightarrow \infty$ as $\| \underline{x} \| \rightarrow \infty$, hence $V \rightarrow \infty$ as $\| \underline{x} \| \rightarrow \infty$ and thus $V(\underline{x})$ is radially unbounded. Here, $\underline{x}$ is the state vector $[x; \dot{x}]$. Summarizing we have,

| | | | |
|---|---|---|---|
| 1) | $V(\underline{x})$ | is positive definite | |
| 2) | Eq. Point | is asymptotically stable | (33) |
| 3) | $V(\underline{x})$ | is radially unbounded. | |

Thus, the equilibrium point is *Globally Asymptotically Stable* for equation 29 under the influence of a Coulomb-type controller.

Figure 5 shows numerical simulation of equation (29) in state space for a linear system under a Coulomb-type controller. For a linear system, $f(x, \dot{x}) = 2\xi\omega\dot{x}$ and $g(x) = \omega^2 x$. The parameters for Figure 5 are $\omega = 40.8$ rad/s, $\xi = 0.25\%$, and $C = 0.5$.

The theorem of total stability presented earlier can again be invoked to show stability of equation (29) under small bounded external perturbations. However, in civil engineering applications, systems modeled with equation (29) are frequently subjected to environmental loads. For example, equation (29) may represent along-wind motion of a structure excited by turbulence in the wind and it might be difficult to argue that the external disturbance due to environmental forces can be treated as small. Hence, it is important to establish boundedness of the solutions of equation (29) under general external disturbance. In general, such an analysis is complicated for a generalized equation as in equation (29). Hence, a linear system with a Coulomb-type controller will be examined under a general external forcing function.

Consider a variation of equation (29) with linear functions for $f(x, \dot{x})$ and $g(x)$,

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = F(t) - Csgn(\dot{x}), \tag{34}$$

where $F(t)$ can be represented as $F(t) = \sum_{i=1}^{N} A_i \cos(\omega_i t)$. A solution of this equation can be written by considering a linear damped equation on the left-hand side being forced by the terms on the right-hand side. Hence,

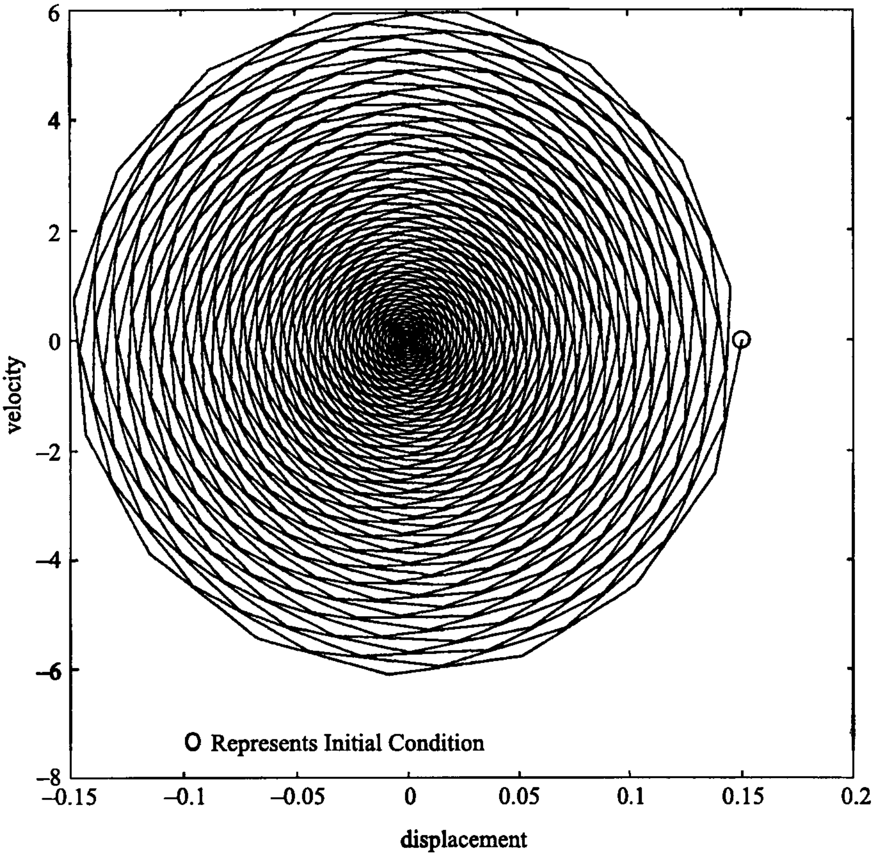


Figure 5. State space of linear system with a Coulomb-type controller.

$$\begin{aligned} x(t) &= \frac{1}{\omega_D}\int_0^t F(\tau)\sin(\omega_D(t-\tau))\exp(-\xi\omega(t-\tau))d\tau \\ &- \frac{1}{\omega_D}\int_0^t Csgn(\dot{x})\sin(\omega_D(t-\tau))\exp(-\xi\omega(t-\tau))d\tau\,, \end{aligned} \tag{35}$$

or, since $|sgn(\dot{x})| \leq 1$,

$$\begin{aligned} x(t) &\leq \frac{C}{\omega_D}\int_0^t \sin(\omega_D(t-\tau))\exp(-\xi\omega(t-\tau))d\tau \\ &- \frac{1}{\omega_D}\int_0^t F(\tau)\sin(\omega_D(t-\tau))\exp(-\xi\omega(t-\tau))d\tau\,. \end{aligned} \tag{36}$$

Since both the integrals on the right-hand side of the preceding inequality are bounded in $[0, \infty)$and as $t \to \infty$ (if $F(t)$ is bounded), hence $x(t)$ is also bounded in $[0, \infty)$ and as $t \to \infty$. The linear form of equation (29) is thus expected to produce stable bounded oscillations under external bounded disturbances of any magnitude.

## 5. CONCLUSIONS

The stability of a general nonlinear system with two different types of controllers has been investigated in this paper. Lypunov's direct method is shown to provide a simple and powerful tool to establish stability properties of these systems. A control algorithm was presented for a general nonlinear system, which is of importance in several applications. Stability of a nonlinear system with a Coulomb-type controller was also studied, which is also encountered frequently in engineering systems. Stability of a linear system with a Coulomb-type controller was demonstrated for a general forcing function. This analysis is expected to be useful for control problems associated with civil engineering structures under environmental loads.

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