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Minimax Design of Robust Controllers for Flexible Systems

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The design of robust time-delay and saturating controllers based on the range of expected variation of uncertain parameters from their nominal values is investigated. A minimax optimization problem is formulated with the objective of minimizing the maximum value of the cost function over the range of the uncertain parameter. By the adoption of the residual energy as the cost function, the optimization problem formulation is simple because it requires only one equation that is used both as the cost function and constraint. To expedite the optimization process, equations are derived for the gradient of the cost and constraint functions with respect to the parameters of the controller. The proposed technique is illustrated on two examples. The first is a spring–mass–dashpot and the second is the two-mass–spring benchmark problem.

I. Introduction

CONTROL of vibratory structures by filtering the reference input to the system has been addressed by numerous researchers including Smith,[1] Singer and Seering,[2] Junkins et al.,[3] Singh and Vadali,[4] and others. Smith[1] proposed a wave cancellation technique to drive a second-order system to its final position in finite time. However, this technique was sensitive to modeling errors. Singer and Seering[2] proposed an approach referred to as input shaping which resulted in the same solution as Smith's.[1] They then proposed a simple technique to desensitize the input shaper to modeling errors.[2] This involved design of a sequence of impulses that forced the magnitude of the residual energy and its derivative with respect to damping or natural frequency to zero. Singh and Vadali[4] arrived at the same results of Singer and Seering[2] by the design of a time-delay filter that canceled the poles of the system. They also showed that by cascading the time-delay filter designed to cancel the poles of the system, the resulting filter was insensitive to errors in modeled damping and frequency.[4] The idea of locating multiple zeros of a time-delay filter at the estimated location of the poles of the system has been exploited to design robust time-optimal control,[5,6] robust fuel-time optimal control,[7] fuel constrained time-optimal control,[8] etc. Liu and Singh[9] extended this idea to nonlinear systems undergoing rest-to-rest maneuver by requiring the sensitivity of the system states with respect to uncertain parameters be zero at the final time.

Techniques to increase the range of uncertain parameters where the residual vibration is below a prespecified amount has been addressed by Singhose et al.[10] This was referred to as the extra insensitive input shaper. Pao et al.[11] proposed including the probability distribution of the uncertain parameters into the design process to arrive at input shapers that weighted the nominal value of the uncertain parameter the most.

The design of time-optimal controllers, extra insensitive controllers, etc., involves formulating an optimization problem with numerous constraints that correspond to satisfaction of the boundary conditions for rest-to-rest maneuvers. For instance, for the design of the time-optimal control profile for the floating oscillator benchmark problem, which is a fourth-order system, four equality constraints need to be satisfied. The formulation of the problem, in addition to being time consuming, results in an optimization problem with numerous equality constraints. In this paper, a simple technique is proposed that uses only one equation, which is referred to as the pseudoenergy of the system. This equation is used both as the objective function and as a constraint equation when it is evaluated at the nominal values of the uncertain parameters of the model. This technique, thus, results in an optimization problem that is easy to formulate. The focus of this paper is on the development of a technique to design time-delay filters that minimize the maximum magnitude of the residual vibration over the range in which the uncertain parameter resides. The resulting controller will be referred to as the minimax time-delay controller. Sections II and III will review the development of the time-delay control and saturating controllers, respectively. This will be followed by the development of the minimax time-delay controller in Sec. IV. The Van Loan[12] identity is used to arrive at equations that represent the gradients of the cost and constraint equations with respect to the parameters of the controller in Sec. V. The proposed technique is illustrated on numerical examples in Sec. VI, and Sec. VII summarizes results generated in this paper.

II. Time-Delay Control

The time-delay control can be considered as a filtering technique that modifies the reference input to a system whose dynamics are characterized by underdamped response (Fig. 1). Singh and Vadali[4] propose a single time-delay filter with a transfer function

$$u(s)/r(s) = A_0/(A_0+1) + e^{-sT}\big/(A_0+1) \tag{1}$$

to minimize the residual vibration of a single-mode underdamped system and show that to cancel a pair of complex conjugate poles located at

$$s = -\zeta\omega \pm j\omega\sqrt{1-\zeta^2} \tag{2}$$

we require

$$A_0 + e^{\zeta\omega T}\cos\left(\omega\sqrt{1-\zeta^2}T\right) = 0 \tag{3}$$

$$e^{\zeta\omega T}\sin\left(\omega\sqrt{1-\zeta^2}T\right) = 0 \tag{4}$$

This results in the solution

$$A_0 = \exp\left(\zeta\pi\big/\sqrt{1-\zeta^2}\right), \qquad T = \pi\big/\omega\sqrt{1-\zeta^2} \tag{5}$$

To address the issue of sensitivity of the pole cancellation time-delay filter, a two time-delay filter is proposed with the constraint that the derivative of the pole cancellation constraint with respect to ζ or ω be forced to zero. The resulting time-delay filter was shown to consist of two single time-delay filters [Eq. (1)], in cascade. This process of cascading a series of single time-delay filters will progressively increase the insensitivity of the filter to modeling errors. However, the penalty of increased settling time of the response of the system can be significant.

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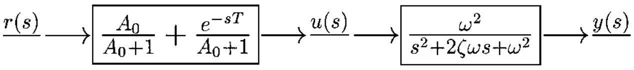

Fig. 1 Single time-delay controlled system.

The design of time-delay filters to cancel two or more pairs of stable complex conjugate poles follows the same procedure outlined earlier. However, the possibility of determining a closed-form solution for the parameters of the time-delay filter with a transfer function

$$u(s)/r(s) = A_0 + A_1 e^{-sT_1} + A_2 e^{-sT_2} + \cdots \tag{6}$$

is remote. To design a multimode time-delay filter, we need to solve a set of nonlinear coupled equation derived by substituting

$$s = -\zeta_i\omega_i + j\omega_i\sqrt{1-\zeta_i^2}, \qquad i = 1, 2, \ldots \tag{7}$$

into Eq. (6) and equating it to zero. The issue of robustness to modeling errors is addressed by cascading time-delay filters designed to cancel the poles of the system, in series.

III. Saturating Controllers

Cost functions such as time, fuel, and weighted fuel-time result in optimal control profiles that are bang–bang or bang–off–bang. These control profiles are very sensitive to uncertainties in modeling, and there is, thus, a need to design controllers that are insensitive to modeling errors. This has been addressed by Liu and Wei,[6] Singhose et al.,[8] Singh and Vadali,[5] and others, where an optimization problem has been formulated that involves design of a time-delay filter that is required to locate multiple zeros of the time-delay filter at the estimated location of the poles of the system. The constraints for the optimization problem are derived by requiring that the boundary conditions for rest-to-rest or spin-up maneuvers be satisfied for the nominal values of the model parameters. Additional constraints that require the sensitivity of the final states to the uncertain parameters be zero are included in the optimization problem. For instance, the transfer function of a time-delay filter for the benchmark floating oscillator (Fig. 2) is

$$1 - 2e^{-sT_1} + 2e^{-sT_2} - 2e^{-sT_3} + e^{-sT_4} \tag{8}$$

The time-optimal control profile is generated by driving the time-delay filter with a step input. The constraints for a rest-to-rest maneuver with zero initial conditions can be shown to be

$$-2T_1 + 2T_2 - 2T_3 + T_4 = 0 \tag{9}$$

$$1 + 2\sum_{i=1}^{3}(-1)^i e^{\zeta\omega T_i}\cos\left(\omega\sqrt{1-\zeta^2}T_i\right) + e^{\zeta\omega T_4}\cos\left(\omega\sqrt{1-\zeta^2}T_4\right) = 0 \tag{10}$$

$$2\sum_{i=1}^{3}(-1)^i e^{\zeta\omega T_i}\sin\left(\omega\sqrt{1-\zeta^2}T_i\right) + e^{\zeta\omega T_4}\sin\left(\omega\sqrt{1-\zeta^2}T_4\right) = 0 \tag{11}$$

$$T_4^2/2 - (T_4 - T_1)^2 + (T_4 - T_2)^2 - (T_4 - T_3)^2 = 2m\theta_f \tag{12}$$

where θ_f indicates the total displacement of the rest-to-rest maneuver.[5] The parameters of the time-delay filter are derived by finding a solution that satisfies all of the constraints and minimizes T_4. To desensitize the controller to modeling errors, additional time delays are added to the filter, and constraints are derived by forcing the derivatives of Eqs. (10) and (11) with respect to ω or ζ to be zero.

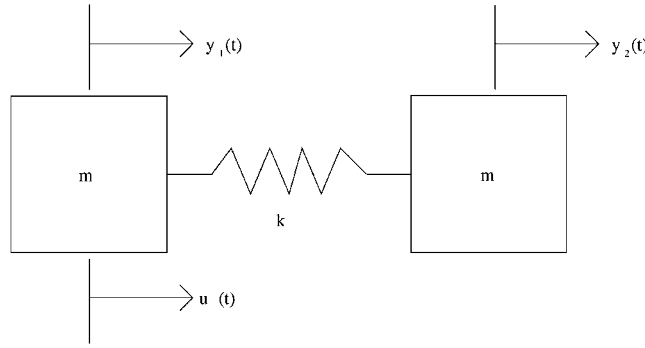

Fig. 2 Two-mass–spring system.

IV. Minimax Time-Delay Control

The time-delay controller and the saturating controllers described earlier are designed using the nominal values of the model parameters. Robustness is arrived at by studying the sensitivity states evaluated at the nominal value of the system parameters. However, with the knowledge that the uncertain parameters lie within a specified range, it is desirable to design a controller with the worst model in mind.[13] This can be achieved by considering the performance of the time-delay filter in the range of the uncertain parameter. In this paper, a design technique is proposed that minimizes the worst performance of the system. The metric used to gauge the performance of the system corresponds to the residual energy of the system at the end of the maneuver. The goal of the optimization problem is to minimize the maximum magnitude of the residual energy in the entire range of the uncertain parameters.

For an asymptotically stable mechanical system undergoing rest-to-rest maneuvers, the model can be represented as

$$M\ddot{y} + C(\boldsymbol{p})\dot{y} + K(\boldsymbol{p})y = Dr \tag{13}$$

where M is a positive-definite matrix and K and C are positive semidefinite. K is positive semidefinite when the model of the system includes rigid-body modes and is positive definite otherwise. Here, $\boldsymbol{p}$ is a vector of uncertain parameters whose elements satisfy the constraints

$$\boldsymbol{p}_i^{\mathrm{lb}} \le \boldsymbol{p}_i \le \boldsymbol{p}_i^{\mathrm{ub}} \tag{14}$$

where $\boldsymbol{p}_i^{\mathrm{lb}}$ and $\boldsymbol{p}_i^{\mathrm{ub}}$ represent the lower and upper bounds on the parameters respectively. The objective here is to design a time-delay filter that prefilters the reference input r to the system with the objective of minimizing the maximum value of the residual energy:

$$\min_x \max_p F$$

$$F = \sqrt{\tfrac{1}{2}\dot{y}^T M\dot{y} + \tfrac{1}{2}(y - y_f)^T K(y - y_f)} \tag{15}$$

where $\boldsymbol{x}$ is a vector of parameters that define the robust time-delay filter, and y_f corresponds to the final displacement states of the system. Equation (15) will be referred to as the pseudoenergy function because it is associated with a hypothetical spring whose potential energy is zero when $y = y_f$. The pseudoenergy function is evaluated at the final time, that is, the end of the maneuver. If K is positive semidefinite, the objective function is

$$\min_x \max_p F$$

$$F = \sqrt{\tfrac{1}{2}\dot{y}^T M\dot{y} + \tfrac{1}{2}(y - y_f)^T K(y - y_f) + 0.5(y_r - y_{rf})^2} \tag{16}$$

where y_r corresponds to the rigid-body displacement and y_{rf} refers to the corresponding desired final displacement. The last term is added to make the cost function positive definite. Note that the parameters of the time-delay filter for systems with a rigid-body mode should be selected to locate two poles at the origin. For an undamped system this is equivalent to requiring the control profile to be antisymmetric about the midmaneuver time.

The preceding formulation weights every point in the uncertain region uniformly. If the designer is provided with information regarding the probability distribution of the uncertain parameters, this information can be included in the optimization problem. For instance, if a Gaussian distribution is assumed for the uncertain parameter, the objective function defined by Eq. (15) can be rewritten as

$$\min_{x} \max_{p} F$$
$$F = \exp\left[-(\boldsymbol{p} - \boldsymbol{p}_{\text{nom}})^T \Gamma^{-1} (\boldsymbol{p} - \boldsymbol{p}_{\text{nom}})\right] \sqrt{\tfrac{1}{2}\dot{y}^T M \dot{y} + \tfrac{1}{2}(y - y_f)^T K (y - y_f)} \tag{17}$$

where Γ is the covariance matrix of the Gaussian distribution and $\boldsymbol{p}_{\text{nom}}$ is a vector of nominal values of the parameters.[11] Without loss of generality, we can assume that the initial displacement states are zero for the study of rest-to-rest maneuvers. We will derive the necessary equations for the optimization problem based on this assumption.

V. Closed-Form Equations

Given a state-space model for a system

$$\dot{z} = Az + Bu, \qquad \text{where} \qquad z = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \in \mathcal{R}^n, \qquad u \in \mathcal{R}^1 \tag{18}$$

where, for systems without rigid-body modes, u is parameterized as

$$u = A_0 + \sum_{i=1}^{n} A_i \mathcal{H}(t - T_i) \tag{19}$$

where $\mathcal{H}(t - T_i)$ is the Heaviside function, or u is

$$u = 1 + \sum_{i=1}^{n} 2(-1)^i \mathcal{H}(t - T_i) + \mathcal{H}(t - T_{n+1}) \tag{20}$$

for a time-optimal controller for systems with rigid-body modes, given that abs(u) is less than 1. Assuming a rest-to-rest maneuver where the initial conditions of the system are zero, the states of the system represented by Eq. (18) can be solved for easily by the technique proposed by Van Loan.[12] To determine the response of a linear system [Eq. (18)] to a unit step input, construct a matrix

$$P = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tag{21}$$

that is an $\mathcal{R}^{n+1 \times n+1}$ matrix. Using the Van Loan identity[12] one can show that

$$Z = e^{PT} = \begin{bmatrix} e^{AT} & \int_0^T e^{A(T-\tau)} B \, \mathrm{d}\tau \\ 0 & I \end{bmatrix} \tag{22}$$

It can be seen that the upper-right-hand term of the matrix Z is the convolution integral of the system given by Eq. (18) subject to a unit step input. Thus, the value of the states at time T for a unit step input are given by the first n rows of the last column of Z. This permits us to calculate the final states for a step input accurately, without numerical integration. This is very attractive for numerical optimization, where a significant cost of optimizing a dynamic system is contributed by the numerical simulation of the response of the system. For instance, the response of the system represented by Eq. (18) to the input represented by Eq. (19) is given by the first n rows of the last column of the matrix

$$\Phi = A_0 e^{PT_n} + \sum_{i=1}^{n} A_i \exp[P(T_n - T_i)] \tag{23}$$

and by the first n rows of the last column of the matrix

$$\Phi = e^{PT_{n+1}} + \sum_{i=1}^{n} 2(-1)^i \exp[P(T_{n+1} - T_i)] \tag{24}$$

for the time-delay control [Eq. (19)] and the time-optimal control [Eq. (20)], respectively.

The optimization algorithms that are used to solve minimax problems are generally gradient based. Thus, the accuracy and the speed of the optimization can be increased by providing analytical gradients to the optimization algorithm. Fortunately, for the time-delay control and the bang–bang and bang–off–bang control profile, closed-form equations representing the gradients of the cost and constraints can be easily derived, as shown hereafter.

For the optimization algorithm, we require the value of the gradient of the cost F and the constraints with respect to the controller parameters. For the control given by Eq. (19), the gradients of F [Eq. (15)] with respect to A_i and T_i are given by

$$\frac{\mathrm{d}F}{\mathrm{d}A_i} = \frac{1}{2\sqrt{\frac{1}{2}\dot{y}^T M \dot{y} + \frac{1}{2}(y - y_f)^T K (y - y_f)}} \times \left[\dot{y}^T M \frac{\mathrm{d}\dot{y}}{\mathrm{d}A_i} + (y - y_f)^T K \frac{\mathrm{d}y}{\mathrm{d}A_i}\right] \tag{25}$$

$$\frac{\mathrm{d}F}{\mathrm{d}T_i} = \frac{1}{2\sqrt{\frac{1}{2}\dot{y}^T M \dot{y} + \frac{1}{2}(y - y_f)^T K (y - y_f)}} \times \left[\dot{y}^T M \frac{\mathrm{d}\dot{y}}{\mathrm{d}T_i} + (y - y_f)^T K \frac{\mathrm{d}y}{\mathrm{d}T_i}\right] \tag{26}$$

To determine $\mathrm{d}y/\mathrm{d}A_i$, $\mathrm{d}\dot{y}/\mathrm{d}A_i$ and $\mathrm{d}y/\mathrm{d}T_i$, $\mathrm{d}\dot{y}/\mathrm{d}T_i$, we require the derivative of the state equation (18). The solution of the equation

$$\frac{\mathrm{d}\dot{z}}{\mathrm{d}A_i} = A\frac{\mathrm{d}z}{\mathrm{d}A_i} + B\mathcal{H}(t - T_i), \qquad i = 0, 1, 2, 3, \ldots \tag{27}$$

can be derived using the Van Loan identity[12] as described earlier. Similarly, the solution of equation

$$\frac{\mathrm{d}\dot{z}}{\mathrm{d}T_i} = A\frac{\mathrm{d}z}{\mathrm{d}T_i} - B[A_i \delta(t - T_i)], \qquad i = 1, 2, 3, \ldots \tag{28}$$

where $\delta(.)$ is the Dirac delta function, can be shown to be

$$\frac{\mathrm{d}z}{\mathrm{d}T_i}(T_f) = -A_i \exp[A(T_f - T_i)]B \tag{29}$$

With the analytical gradients, we can expedite the convergence of the optimization algorithm. Next, for the control given by Eq. (20), the gradients of F [Eq. (15)] with respect to T_i are given by Eq. (26), where

$$\frac{\mathrm{d}z}{\mathrm{d}T_i}(T_f) = -2(-1)^i \exp[A(T_f - T_i)]B, \qquad i = 2, 3, \ldots, n-1 \tag{30}$$

The analytical gradients can also be calculated as follows: The state response for the system

$$\dot{z} = Az + B\sum_{i=1}^{N} A_i \mathcal{H}(t - T_i) \tag{31}$$

can be represented as

$$z(t) = \sum_{i=1}^{N} A_i \exp(At)[\exp(-AT_i) - \exp(-At)]A^{-1}B \tag{32}$$

The sensitivity of the states to the variables A_i and T_i can be calculated from Eq. (32). For instance, the sensitivity of the states to the final time ($T_f = T_N$) is

$$\frac{\mathrm{d}z}{\mathrm{d}T_N}(T_N) = \sum_{i=1}^{N-1} A_i \exp[A(T_f - T_i)]B \tag{33}$$

VI. Numerical Examples

A. Spring–Mass–Dashpot

The proposed technique will be illustrated on a rest-to-rest maneuver of a single-mode system whose dynamics are defined by the equation

$$m\ddot{y} + c\dot{y} + ky = kr \tag{34}$$

with the boundary conditions

$$y(0) = \dot{y}(0) = 0, \qquad y(t_f) = 1, \qquad \dot{y}(t_f) = 0 \tag{35}$$

where t_f is the maneuver time.

First, a minimax time-delay controller will be designed, assuming that only k is uncertain and satisfies the constraint

$$0.7 \le k \le 1.3 \tag{36}$$

where the nominal value of $k = 1$, $m = 1$, and $c = 0.2$. The form of the transfer function for the minimax time-delay controller is chosen to be

$$A_0 + A_1 e^{-sT_1} + A_2 e^{-sT_2} \tag{37}$$

which is identical to the robust time-delay controller.[4] The optimization problem can be stated as the determination of A_0, A_1, A_2, T_1, and T_2 of the time-delay filter to

$$\min_{T_i, A_i} \max_{k} \sqrt{\tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}k(y-1)^2} \tag{38}$$

evaluated at T_2. The initial guess for the minimax optimization problem is the robust time-delay filter. To determine the parameters of the robust time-delay filter, we need to solve for the nonrobust time-delay filter first. The transfer function of the nonrobust time-delay filter for the nominal system is

$$0.5783 + 0.4217e^{-3.1574s} \tag{39}$$

The minimax problem for the nonrobust time-delay filter was solved using the closed-form equation derived in this paper, as well as a standard differential equation integrator, to determine the terminal states for the optimizer. Table 1 illustrates the improvement in convergence of the optimizer.

With the knowledge that two nonrobust filters in cascade will force the derivative of the square root of the pseudoenergy to be zero at the nominal value of the system parameters, resulting in smaller magnitude of residual vibration in the vicinity of the nominal parameters [as illustrated in Fig. 3 (dashed line)], the transfer function of the robust time-delay controller can be shown to be

$$0.3344 + 0.4877e^{-3.1574s} + 0.1788e^{-6.3148s} \tag{40}$$

The parameters of the time-delay filter [Eq. (40)], will be used as initial guesses for the minimax algorithm. The optimization toolbox of MATLAB® is used to solve the minimax optimization problem. The optimal minimax time-delay filter is given by the transfer function

$$0.3452 + 0.4730e^{-3.1703s} + 0.1818e^{-6.2060s} \tag{41}$$

Figure 3 (dotted line) illustrates the variation of the residual energy of the system as a function of the uncertain parameter k. It can be seen that the maximum magnitude of the residual energy in the range of the uncertain parameters occurs at the bounding limits, $k = 0.7$, $k = 1.3$, and at a value of k that lies between the limits. It is also clear that the maximum magnitude of the residual energy is significantly smaller than that resulting from the robust time-delay filter defined by Eq. (40) over the entire range of k. However, at the nominal value of $k = 1$, the minimax solution has a large magnitude of residual vibration. The minimax solution is similar to the extra insensitive input shaper proposed by Singhose et al.,[10] where an optimization problem is formulated by defining the magnitude of residual vibration permitted at the nominal value of the uncertain parameter and solving for the magnitudes of a sequence of impulses.

Table 1 Performance of optimizer

Optimizer	Time of convergence, s	No. of floating-point operations
Proposed approach	1.3071	687,148
Numerical integration	37.8023	5,665,488

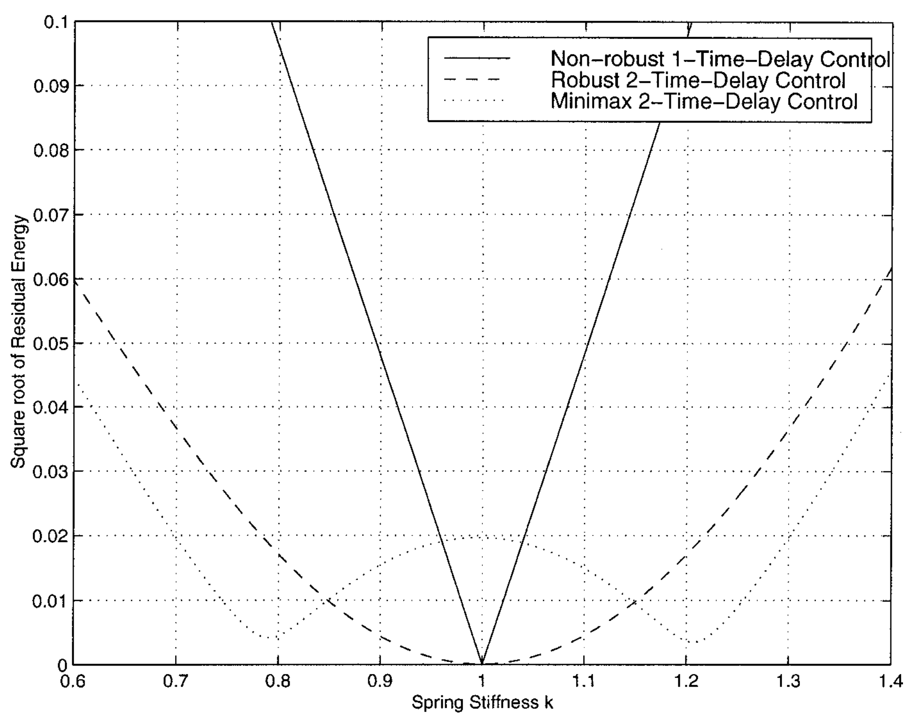

Fig. 3 Residual vibration distribution.

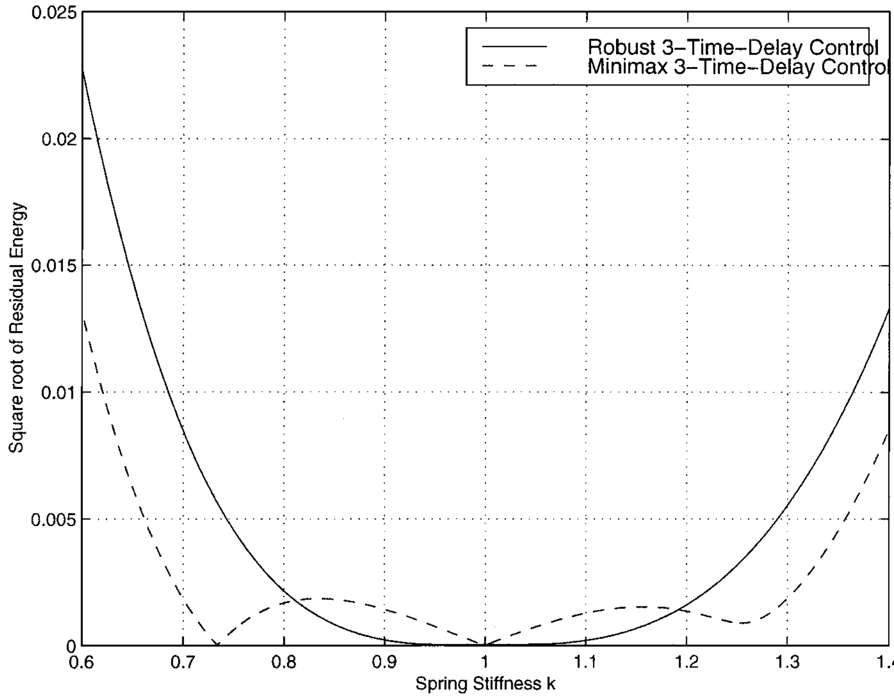

Fig. 4 Residual vibration distribution.

The impulse sequence is required to satisfy the constraints that the magnitude of the residual vibration is zero at two frequencies that flank the nominal value, and the slope of the residual energy distribution curve is zero at the nominal value of the uncertain parameter. These constraints are unnecessary because, as is shown in Fig. 3, the minimax solution does not force the residual energy curve to zero in the range of uncertain parameters.

Notwithstanding that the maximum magnitude of the residual vibration over the range of possible value of k has been minimized, that the residual vibration at the nominal value of $k = 1$ is large is a drawback of this controller. To address the aforementioned disadvantage, an additional constraint is included into the minimax optimization problem that requires the magnitude of the residual vibration to be zero at the nominal value of the uncertain parameter. The added constraint necessitates addition of a time delay to the time-delay filter defined by Eq. (41), resulting in the transfer function

$$A_0 + A_1 e^{-sT_1} + A_2 e^{-sT_2} + A_3 e^{-sT_3} \tag{42}$$

The unknown parameters of Eq. (42) are solved for using the solution of the parameters of three nonrobust time-delay filters in cascade as the initial guess. The transfer function of the minimax time-delay controller with the constraint to force the residual vibration to be zero at the nominal value of k can be shown to be

$$0.2052 + 0.4141 e^{-3.1652s} + 0.3015 e^{-6.3304s} + 0.07924 e^{-9.4956s} \tag{43}$$

Figure 4 illustrates the distribution of the residual energy of the time-delay filter designed by cascading three nonrobust time-delay filters (solid line) and the minimax time-delay filter (dashed line). It is clear from Fig. 4 that the maximum magnitude of the residual energy of the minimax controller over the uncertain range ($0.7 \le k \le 1.3$) is significantly smaller than the robust three-time-delay controller, which is a metric to gauge the robustness of the controllers.

B. Floating Oscillator

The second example considered for the illustration of the proposed technique is the benchmark two-mass–spring system illustrated in Fig. 2. Unlike the first example, this system is characterized by rigid-body modes, and the sum of the kinetic and potential energy is not a positive-definite function. Therefore, the energy of the system is augmented with a term that reflects the energy stored in a virtual spring whose potential energy is zero when the masses are at the final desired positions. The equations of motion of the floating oscillator are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{44}$$

where u is bounded by the constraint

$$-1 \le u \le 1 \tag{45}$$

The objective of the optimization problem is to design a control profile for a rest-to-rest maneuver that satisfies the boundary conditions

$$\begin{gathered} y_1(0) = y_2(0) = \dot{y}_1(0) = \dot{y}_2(0) = 0 \\ y_1(t_f) = y_2(t_f) = 1, \qquad \dot{y}_1(t_f) = \dot{y}_2(t_f) = 0 \end{gathered} \tag{46}$$

The antisymmetric optimal control profile is parameterized as

$$\begin{aligned} u = {} & 1 + \sum_{i=1}^{n} 2(-1)^i \mathcal{H}(t - T_i) + 2(-1)^{n+1} \mathcal{H}(t - T_{n+1}) \\ & + \sum_{i=1}^{n} 2(-1)^i \mathcal{H}(t - (2T_{n+1} - T_i)) + \mathcal{H}(t - 2T_{n+1}) \end{aligned} \tag{47}$$

A minimax problem is formulated to solve for the maneuver and switch times T_i, assuming that k is uncertain and satisfies the constraint

$$0.7 \le k \le 1.3 \tag{48}$$

where the cost function is

$$\begin{aligned} \min_{T_i} \max_{k} & \left[\frac{1}{2} [\dot{y}_1 \quad \dot{y}_2] \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \right. \\ & \left. + [y_1 \quad y_2] \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{1}{2} (y_1 - 1)^2 \right]^{0.5} \end{aligned} \tag{49}$$

Assuming $n = 1$ in Eq. (47), and solving the minimax problem with the constraint that the magnitude of the pseudoenergy function [Eq. (16)] be zero at $k = 1$ (the nominal value of the uncertain

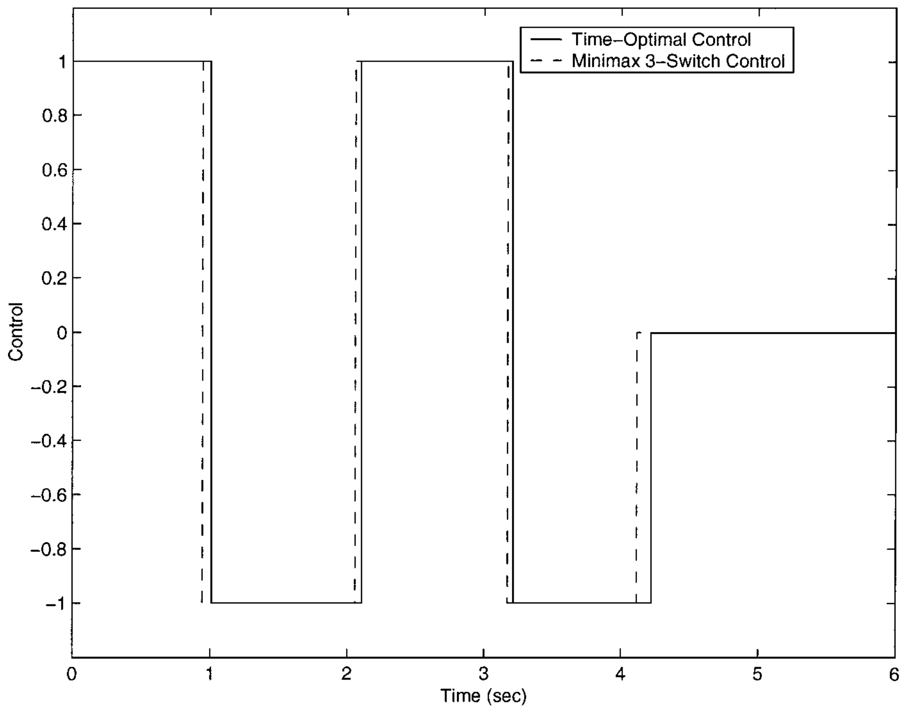

Fig. 5 Optimal control profile.

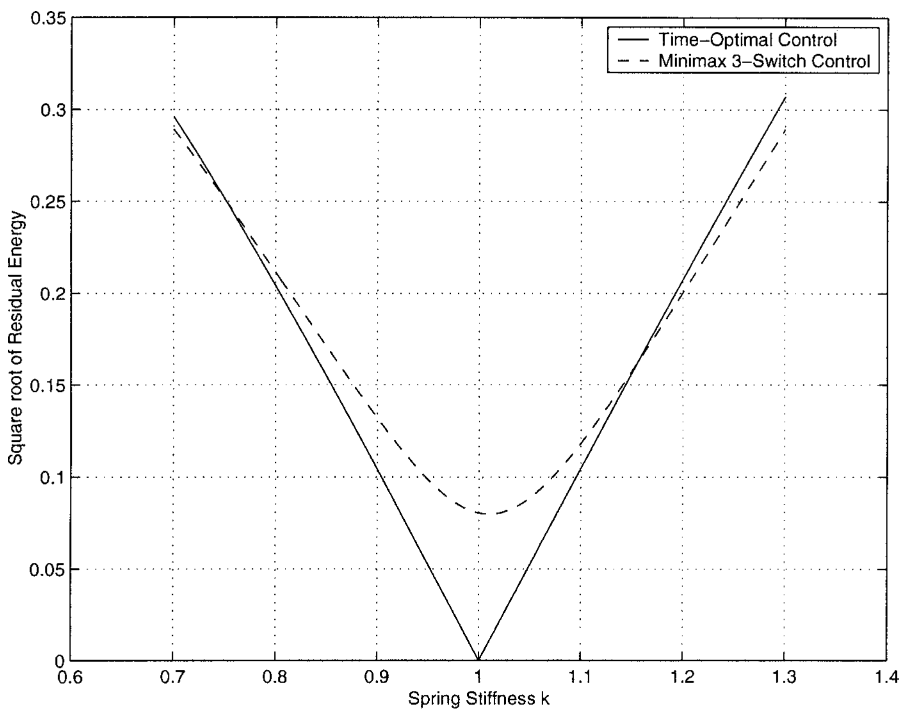

Fig. 6 Residual energy distribution.

parameter), we arrive at the time-optimal control profile

$$u = 1 - 2\mathcal{H}(t - 1.0027) + 2\mathcal{H}(t - 2.1089)$$

$$- 2\mathcal{H}(t - 3.2151) + \mathcal{H}(t - 4.2178) \tag{50}$$

Solving the minimax control problem without enforcing the requirement that the residual energy should be zero at the nominal value of the uncertain parameter results in the control profile

$$u = 1 - 2\mathcal{H}(t - 0.9430) + 2\mathcal{H}(t - 2.0571)$$

$$- 2\mathcal{H}(t - 3.1713) + \mathcal{H}(t - 4.1143) \tag{51}$$

Figure 5 illustrates the control profile for the three-switch time-optimal (solid line) and minimax control profile (dashed line). Figure 6 illustrates the variation of the residual energy of the floating oscillator as a function of k for the three-switch control profiles. It is clear that the maximum magnitude of the residual energy of the control profile given by Eq. (51) (dashed line) is smaller than that given by Eq. (50) in the range of uncertain k. Note that the maximum magnitude of the residual vibration occurs at the end of the interval of uncertainty, that is, $k, = 0.7$ and $k = 1.3$. However, the residual energy is not zero at $k = 1$.

A five-switch control profile is selected next, to reduce the maximum magnitude of residual energy. The minimax optimization problem is solved again, with and without the constraint that the residual energy for $k = 1$ should be zero. The resulting control

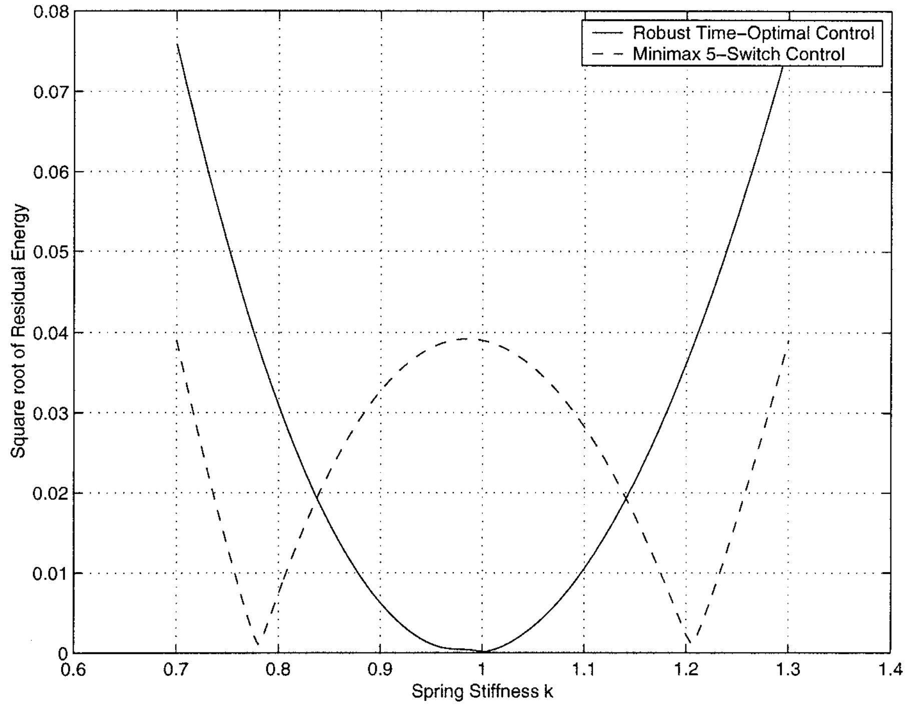

Fig. 7 Residual energy distribution.

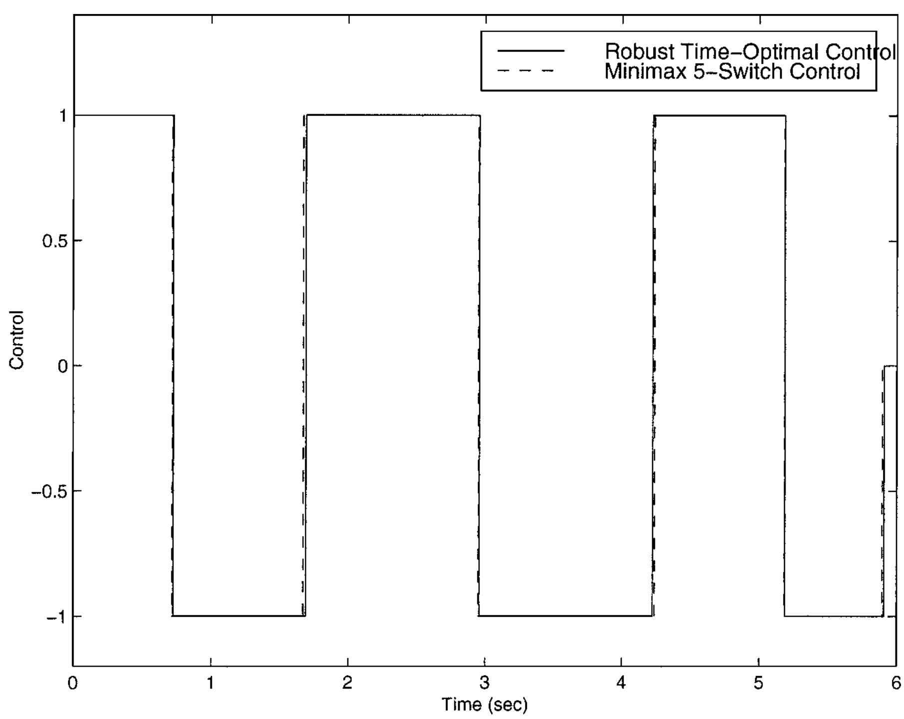

Fig. 8 Optimal control profile.

profile is

$$u = 1 - 2\mathcal{H}(t - 0.7181) + 2\mathcal{H}(t - 1.6715) - 2\mathcal{H}(t - 2.9526)$$

$$+ 2\mathcal{H}(t - 4.2370) - 2\mathcal{H}(t - 5.1851) + \mathcal{H}(t - 5.8944) \quad (52)$$

The solid line of Fig. 7 illustrates the distribution of the residual energy with constraint that the residual energy be zero at $k = 1$. Note that the optimization algorithm resulted in a control profile that forces the slope of the energy distribution curve to be zero at $k = 1$, without the explicit requirement of that constraint as in the work of Singh and Vadali.[5] The dashed line of Fig. 7 illustrates that the elimination of the constraint that the residual energy be zero at $k = 1$ results in a significant reduction of the maximum magnitude of residual energy in the uncertain ranges, which again occur at the ends of the uncertain region. This residual energy distribution is similar to the one presented by Singhose et al.[10] Note that the residual energy is not zero at the nominal value of $k = 1$. The control profile is given by the equation

$$u = 1 - 2\mathcal{H}(t - 0.7256) + 2\mathcal{H}(t - 1.6909) - 2\mathcal{H}(t - 2.9595)$$

$$+ 2\mathcal{H}(t - 4.2281) - 2\mathcal{H}(t - 5.1934) + \mathcal{H}(t - 5.9190) \quad (53)$$

Figure 8 illustrates the control profile for the five-switch time-optimal (solid line) and minimax control profile (dashed line).

In both of the examples presented in this section, optimization problems were solved with the evaluation of the cost function at 3, 5, 11, 41, and 81 discrete points in the uncertain region. All of them resulted in the same solution.

VII. Conclusions

This paper presents a technique for the design of robust controllers that minimize the maximum magnitude of the cost function over the uncertain interval. The proposed cost function is the residual energy of a system that includes the kinetic and potential energy. For systems with rigid-body modes, a pseudoenergy term is added to create a positive-definite energy function. A simple technique is proposed for the evaluation of the cost function and constraints without numerical simulation of the state equations. Furthermore, closed-form equations for the gradients of the cost and constraint functions with respect to the parameters to be optimized for are derived, which aid in the numerical optimization. The proposed technique is illustrated on two examples. The first is a spring–mass–dashpot and involves the design of a time-delay filter to minimize the maximum magnitude of residual vibration for a unit step input. The second example is the design of a robust bang–bang controller for rest-to-rest maneuvers of the two-mass–spring benchmark problem.

The proposed technique has been illustrated for systems with one uncertain parameter. It can easily be extended to design time-delay filters that are insensitive to multiple parameters. This entails determining the residual energy at points in the uncertain space that are uniformly sampled, to create a vector of costs for the minimax algorithm. The equations derived in Sec. V are also valid for multiple parameters.

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