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# ENHANCED CONVERGENCE IN DISTRIBUTED DESIGN PROCESSES

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## ABSTRACT

*The focus of this paper is on studying the convergence properties of the solution process of decentralized or distributed subsystems, where each subsystem has its own design problem, including objective(s), constraints, and design variables. The challenging aspect of this type of problem comes in the coupling of the subsystems, which create complex research and implementation challenges in modeling and solving these types of problems. We focus on the dynamics of these distributed design problems and attempt to further the understanding of the fundamental mechanics behind these processes in order to support the decisions being made by a network of decision makers. In this work, the domain of attraction, or region where convergence to a stable equilibrium point is guaranteed, of a decentralized design process is studied. Two approaches based on concepts from nonlinear control theory are presented: the first determines the domain of attraction for a specified Lyapunov function and the second optimizes for a Lyapunov function which maximizes the domain of attraction. The two techniques are illustrated on a benchmark pressure vessel design problem.*

**Keywords:** Decentralized Design, Lyapunov Theory, Semidefinite Programming, Sum of Squares.

## 1 INTRODUCTION

The complexity of system design problems is one reason for the decomposition and decentralization of decisions. For example, The Economist reports that it took 700 parts to make the model Ford T, while modern cars pack many more in their radio alone [1]. From the aerospace industry, there are 3 million parts in a Boeing 777 provided by more than 900 suppliers [2]. Another reason for the decomposition of systems into smaller coupled subproblems is the multidisciplinary nature of these systems, which makes it impossible for one designer, or even a single design team, to consider the entire system as a single design problem. Typically, in complex systems, breaking it up into smaller units or subsystems will make the system more manageable [3, 4]. This decentralization of decisions is unavoidable in a large organization where having only one centralized decision maker is usually not applicable [5]. In fact, decentralization is recommended as a way to speed up product development processes and decrease the computational time and the complexity of the problem [6]. As an example, Airbus first decomposes along the main sections of the airplane, and is then further decomposed into smaller disciplinary subsystems as it is multidisciplinary along "Centres of Excellence" and "Centres of Competence" [7].

While the decomposition of complex problems certainly creates a series of smaller, less complex problems, it also creates several challenging issues associated with the coordination, convergence, and stability of these less complex problems. Previous work has been done on the coordination of decomposed problems using Design Structure Matrices [8], hierarchical approaches [9], or by effectively propagating the desirable top level design specifications to appropriate subsystems [10, 11]. The ef-

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ficiency of these approaches is compared in [12]. Another set of techniques that have been used extensively to solve this kind of problem include methods and constructs from game theory. Previous work in Game Theory includes work to model the interactions between the designers if several design variables are shared among designers [13]. In [14], Game Theory is formally presented as a method to help designers make strategic decisions in a scientific way. In [15], distributed collaborative design is viewed as a non-cooperative game, and maintenance considerations are introduced into a design problem using concepts from Game Theory. In [16], the manufacturability of multi-agent process planning systems is studied using Game Theory concepts. In [17], non-cooperative protocols are studied and the application of Stackelberg leader/follower solutions is shown. Also in [18], a Game Theory approach is used to address and describe a multifunctional team approach for concurrent parametric design. This set of work on game theory in complex systems design studied a set of valuable research issues, but did not formally address the mechanisms of convergence in a generic decentralized design problem. Issues of convergence are critical to study and resolve in the context of these kinds of systems design problems [19], especially when decomposed problems are solved asynchronously [20].

The issue of convergence (and divergence) was introduced in [21], and then studied formally in [22] using geometric series theory for some basic decentralized problems marked by quadratic and unconstrained problems. This work was then expanded in [23] to nonlinear, constrained problems using concepts from linear and nonlinear control theory. An important concept that has emerged from these studies is the idea of a "*domain of attraction*" of an equilibrium point, or a region where if the distributed process starts, the entire process will converge to an equilibrium solution point. These domains of attractions become critical when the decomposed problems are nonlinear and therefore contain multiple equilibrium solution points with different design and stability properties [23]. Further, for processes that are marked by potentially divergent behavior, identifying a robust domain of attraction would be very attractive in systems coordination and solution.

In [23], the domain of attraction was approximated by using Lyapunov functions. A Lyapunov function, $V$, is a positive definite scalar function. If the time derivative of this function is negative definite in a domain $D$, then the system is said to be stable at the origin and $D$ its domain of attraction. This is explained in detail in section 3. In [23], the equilibrium point of a given decentralized design problem is first determined and then moved to the origin whose domain of attraction is estimated by determining the region where the Lyapunov function constraints are satisfied. These domains, while identified for the first time in the context of systems design problems, were limited in their range. Therefore, in this paper, we develop new techniques similar to the one used by [24] to expand both the science of modeling and identifying domains of attraction, and the application to design problems marked by coupled decomposition. Some of the operating assumptions of this work include:

1. The focus is on complex engineering systems, or those systems that necessitate the decomposition of the system into smaller subsystems in order to reduce the complexity of the design problems that must be solved.
2. The decomposed problems are "self-sufficient" in that they contain their own set of independent design variables that they control, their own design objective(s), and limitations on the design in the form of constraints.
3. These decomposed problems can be represented in mathematical formulations.
4. The coupling between the decomposed problems occurs through the design variables. That is the values of the design variables of one decomposed problem are necessary in at least one of the other decomposed problems.

In section 2, we present the basic background for the fundamental model we develop of decentralized design problems: game theory and discrete time series. In section 3, the concept of domain of attraction for a discrete time system is described and Lyapunov-based techniques for the determination of the domain of attraction are presented. Two approaches are presented: the first determines the domain of attraction for a specified Lyapunov function. The second solves for an optimal Lyapunov function that maximizes the provable domain of attraction. In section 4, the proposed techniques are illustrated on a benchmark pressure vessel design problem. Section 5 includes some conclusions and directions for future work.

## 2 BACKGROUND

Table 1 from [23] presents the Game-Theoretic formulation for an optimization design problem with two subsystems where $\mathbf{x}_1$ and $\mathbf{x}_2$ represent the vectors of design variables controlled by subsystems 1 and 2 respectively. $\mathbf{x}_{1c}$ and $\mathbf{x}_{2c}$ are the *non-local* design variables, or variables that appear in a model but are controlled by the other subsystem. In a non-cooperative protocol, players make decisions by assuming the choices of other decision-makers. A solution $(x_{1N}, x_{2N})$ is a Nash solution if

$$\begin{aligned} & F_1(x_{1N}, x_{2N}) = \min_{x_1} F_1(x_1, x_{2N}) \\ \text{and} \quad & F_2(x_{1N}, x_{2N}) = \min_{x_2} F_2(x_{1N}, x_2) \end{aligned} \tag{1}$$

The Nash Equilibrium is the fixed point of two subsets of

| **Player 1's Model:** | | **Player 2's Model** | |
|---|---|---|---|
| *Minimize* | | *Minimize* | |
| $\mathbf{F}_1(\mathbf{x}_1, \mathbf{x}_{2c})$ | | $\mathbf{F}_2(\mathbf{x}_2, \mathbf{x}_{1c})$ | |
| *subject to* | | *subject to* | |
| $g_j^1(\mathbf{x}_1, \mathbf{x}_{2c}) \leq 0$ | $j = 1..m_1$ | $g_j^2(\mathbf{x}_2, \mathbf{x}_{1c}) \leq 0$ | $j = 1..m_2$ |
| $h_k^1(\mathbf{x}_1, \mathbf{x}_{2c}) = 0$ | $k = 1..l_1$ | $h_k^2(\mathbf{x}_2, \mathbf{x}_{1c}) = 0$ | $k = 1..l_2$ |
| $\mathbf{x}_{1L} \leq \mathbf{x}_1 \leq \mathbf{x}_{1U}$ | | $\mathbf{x}_{2L} \leq \mathbf{x}_2 \leq \mathbf{x}_{2U}$ | |

Table 1. MULTI-PLAYER OPTIMIZATION PROBLEM FORMULATION

the feasible space:

$$(x_{1N}, x_{2N}) \in X_{1N}(x_{2N}) \times X_{2N}(x_{1N}) \quad (2)$$
$$\text{where} \begin{cases} X_{1N}(x_2) = \{x_{1N} \,|\, F_1(x_{1N}, x_2) = \min_{x_1} F_1(x_1, x_2)\} \\ X_{2N}(x_1) = \{x_{2N} \,|\, F_2(x_1, x_{2N}) = \min_{x_2} F_2(x_1, x_2)\} \end{cases}$$

$X_{1N}(x_2)$ and $X_{2N}(x_1)$ are called the *Rational Reaction Sets* of the two subsystems.

Our basic model of a decentralized design problem is based on the assumption that the decision makers in the system make new decisions based on information from the previous iteration. At each iteration, each decision maker makes a decision that is consistent with their rational reaction set. That is, they return to their RRS, which minimizes their objective function. This series of iterative decisions among the decision makers can be modeled as a discrete time control problem as given in Eq. (3)

$$x((k+1)) = f(x(k)) \quad (3)$$

In Eq. (3), as explained in [23], $f$ is a representation of the RRS of each subsystem (exact or an approximation). This model dictates that the new values of the design variables, x, are a function of the RRS representation of the information from the previous iteration. If the decentralized systems converges to a solution, then it occurs at the intersection of the subsystem RRS's, which is an equilibrium point as shown in Eq. (2). There can be more than one equilibrium point or none depending on the problem. Given an equilibrium point, its domain of attraction provides valuable information. The *domain of attraction* of an equilibrium point, in the context of decentralized design, is the set of all points from which if the design process is initiated, the solution will converge to the equilibrium point. Thus obtaining a good estimate of the domain of attraction is very important. The following section describes methods to arrive at better estimates of domains of attraction.

## 3 ESTIMATING THE DOMAIN OF ATTRACTION

Consider the model of a dynamic system in discrete time given in Eq. (3) and assume that the origin is the point of equilibrium. The domain of attraction $\mathcal{D}$ is defined as:

$$\mathcal{D} = \{x(0) \in \mathcal{R}^n | x(t, x(0)) \to 0 \text{ as } t \to \infty\} \quad (4)$$

A Lyapunov function $V$ is one which satisfies the constraints [25]:

$$V(0) = 0, V(x(k)) > 0 \text{ for } \mathcal{D} - \{0\} \quad (5)$$
$$\Delta V = V(x(k+1)) - V(x(k)) \leq 0 \text{ in } \mathcal{D}. \quad (6)$$

This implies that the equilibrium point at the origin is stable. Further if

$$\Delta V < 0 \text{ in } \mathcal{D}, \quad (7)$$

the origin is asymptotically stable. If $\Delta V < 0$, within a Lyapunov surface (level set) $V = \beta$, the state trajectory (the evolution of states with time) will pierce Lyapunov surfaces with smaller $\beta$ over time, which implies that the states will never leave the region $V \leq \beta$. It is clear that $V < \beta$ represents a subset of the domain of attraction and maximizing $\beta$ gives the optimal estimate of the domain of attraction for the Lyapunov function $V$.

Seiler [26] proposed a technique using Sum of Squares (SOS) programming to determine the provable domain of attraction of the origin (equilibrium point) given a Lyapunov function for a continuous-time system. For any Lyapunov function $V$, a region $V \leq \beta$ is determined wherein $\dot{V} < 0$ and hence belongs to the domain of attraction. The following method to determine provable domain of attraction was originally proposed by [26] for continuous time systems. In the next section, we present a modified version for discrete time applications.

### 3.1 Zero Detection Algorithm

Given a Lyapunov function $V$, which shows that the origin is stable, consider the problem of estimating the domain of attraction of the origin. The problem can be formulated as

$$\max \gamma$$

subject to

$$\Delta V = V(x(k+1)) - V(x(k)) \leq 0 \text{ when } V \leq \gamma \quad (8a)$$

Consider the implication

$$\Delta V = 0 \Rightarrow x = 0 \text{ or } V \geq \gamma$$

This proves that $\Delta V \neq 0$ in the region $V < \gamma$, except at the origin. Since $\Delta V$ is a continuous function, $\Delta V < 0$ in the region $V < \gamma$. Using the Positivstellensatz Theorem [27], the implication is true if and only if there exists a polynomial $r(x)$ such that

$$(V-\gamma)x^T x + r(x)\Delta V \geq 0. \tag{9}$$

Thus the problem of estimating the domain of attraction, given a Lyapunov function can be reformulated as follows:

$$\max_{r(x)} \gamma$$

subject to

$$(V-\gamma)x^T x + r(x)\Delta V \geq 0 \tag{10a}$$

While this approach will find accurate estimates of the domains of attraction, it does not optimize the Lyapunov function to maximize the domain of attraction. In the next section, an expanding interior algorithm, originally proposed by [24], is introduced for the type of discrete time systems present in decentralized design processes.

### 3.2 Expanding Interior Algorithm for Discrete Time

The expanding interior algorithm maximizes level sets of a predefined closed curve such as a circle which is referred to as a *shape function*, instead of directly maximizing level sets of the Lyapunov function (V). The time derivative of the Lyapunov function, $\dot{V}$ is constrained to be negative definite within the region V = 1 and by maximizing the level sets of the shape function that can be contained within this region, V=1, the algorithm indirectly maximizes the area of the curve V=1.

When $V$ is positive definite,

$$\Omega = \{x \in \mathcal{R}^n | V(k) \leq 1\} \tag{11}$$

is bounded and when

$$\{x \in \mathcal{R}^n | V(k) \leq 1\} \subseteq \{x \in \mathcal{R}^n | \Delta V \leq 0\} \tag{12}$$

$\Omega$ is a subset of the domain of attraction for the system. To enlarge $\Omega$, a variable sized region $P_\beta \in \Omega$ is defined as follows:

$$P_\beta = \{x \in \mathcal{R}^n | p(x(k)) \leq \beta\} \tag{13}$$

where $p(x(k))$ is a positive definite polynomial called the shape function. The problem of improving the domain of attraction can then be reformulated as an optimization problem

$$\max_{V \in \mathcal{R}_n, V(0)=0} \beta$$

subject to

$$\{x \in \mathcal{R}^n | V(k) \leq 0, x \neq 0\} = \phi$$
$$\{x \in \mathcal{R}^n | p(x(k)) \leq \beta, V(k) \geq 1, V(k) \neq 1\} = \phi$$
$$\{x \in \mathcal{R}^n | V(k) \leq 1, \Delta V(k) \geq 0, x \neq 0\} = \phi$$

where $\phi$ represents the null set. To reformulate the problem in way that it can be solved by SOS programming, the constraint $x \neq 0$ is replaced with $l_i(x) \neq 0$, where $l_i(x) \neq 0$ is an SOS. Thus $l_i(x) = 0$ if and only if $x = 0$. Applying the Positivstellensatz theorem the problem becomes

$$\max_{\substack{V \in \mathcal{R}_n, V(0)=0, k_1,k_2,k_3 \in \mathcal{Z}_+ \\ s_1,\ldots,s_{10}, l_1, l_2 \in \Sigma_n}} \beta$$

subject to

$$\begin{aligned} & s_1 - V s_2 + l_1^{2k_1} = 0 \\ & s_3 + (\beta - p)s_4 + (V-1)s_5 \\ & + (\beta - p)(V-1)s_6 + (V-1)^{2k_2} = 0 \\ & s_7 + (1-V)s_8 + \Delta V s_9 + (1-V)\Delta V s_{10} + (l_2)^{2k_3} = 0 \end{aligned} \tag{15a}$$

The problem can be reduced to a much simpler form by picking convenient $k_i$s and $s_i$s. Picking $k_1 = k_2 = k_3 = 1$, $s_2 = l_1$, $s_3 = s_4 = s_{10} = 0$ and simplifying we get,

$$\max_{V \in \mathcal{R}_n, V(0)=0, s_6, s_8, s_9 \in \Sigma_n} \beta$$

subject to

$$\begin{aligned} & V - l_1 = s_1 \\ & -((\beta - p)s_6 + (V-1)) = s_5 \\ & -((1-V)s_8 + \Delta V s_9 + l_2) = s_7 \end{aligned}$$

Or alternatively the problem becomes

$$\max_{V \in \mathcal{R}_n, V(0)=0, s_6, s_8, s_9 \in \Sigma_n} \beta$$

subject to

$$\begin{aligned} & V - l_1 \in \Sigma_n \\ & -((\beta - p)s_6 + (V-1)) \in \Sigma_n \\ & -((1-V)s_8 + \Delta V s_9 + l_2) \in \Sigma_n \end{aligned}$$

Since the above formulation has products of unknowns, it cannot be solved directly by SOS programming using SOSTOOLS [28] (a MATLAB toolbox for solving sum of squares programs). An iterative approach to solving the above problem is as follows:

**Algorithm**

The following algorithm is a modification of the approach presented by Jarvis-Wloszek [24]. The problem formulation by Jarvis-Wloszek [24] is for continuous time systems. The algorithm presented here is modified for discrete time systems. Let $i$ be the iteration index and set $i = 1$. Denote the candidate Lyapunov function $V^{(i=0)}$ and pick the maximum degree of the Lyapunov function, the SOS multipliers and the $l$ polynomials to be $d_V, d_{s_6}, d_{s_8}, d_{s_9}, d_{l_1}, d_{l_2}$ respectively. Set $\beta^{(i=0)} = 0$

1. Set $V = V^{i-1}$ and solve the linesearch on $\beta$

$$\max_{V \in \mathcal{R}_n, V(0)=0, s_6, s_8, s_9 \in \Sigma_n} J = \beta$$
$$\text{subject to} - ((\beta - p)s_6 + (V-1)) \in \Sigma_n$$
$$- ((1-V)s_8 + \Delta V s_9 + l_2) \in \Sigma_n$$

Set $s_8^{(i)} = s_8$ and $s_9^{(i)} = s_9$. Continue to step 2.

2. Set $s_8 = s_8^{(i)}$ and $s_9 = s_9^{(i)}$ and solve the linesearch on $\beta$

$$\max_{V \in \mathcal{R}_n, V(0)=0, s_6, s_8, s_9 \in \Sigma_n} J = \beta$$

subject to

$$V - l_1 \in \Sigma_n$$
$$- ((\beta - p)s_6 + (V-1)) \in \Sigma_n$$
$$- ((1-V)s_8 + \Delta V s_9 + l_2) \in \Sigma_n$$

Set $\beta^{(i)} = \beta$ and $V^{(i)} = V$. If $\beta^{(i)} - \beta^{(i-1)}$ is less than a specified tolerance go to step 3, else increment $i$ and go to step 1.

3. The set $\{x \in R^n | V^{(i)}(x) \leq 1\}$ contains $p(x) = \beta$ and is the largest estimate of the fixed points region of attraction.

This algorithm is implemented in the next section on a two subsystem design problem.

## 4 EXAMPLE

In this section, the benchmark thin-walled pressure vessel design with hemispherical ends, shown in Figure 1, is considered to illustrate the two approaches that have been presented in section 3. The nomenclature for this case study is taken from [23] and is presented in Table 2.

The multi-objective problem is to minimize the weight and maximize the volume of the cylinder, subject to geometric and stress constraints. The internal pressure $P$ and the material of the vessel are specified. The problem is assumed to involve two interacting design teams, one who controls the volume by manipulating the radius $R$ and the length $L$, and the second who controls the weight via the variable, thickness $T$. The cost and constraints for the two teams are listed in Tables 3 and 4. The rational reaction sets of each subsystem can either be found analytically as in [17], or approximated as in [23]. The RRS approximations from [23] are shown below:

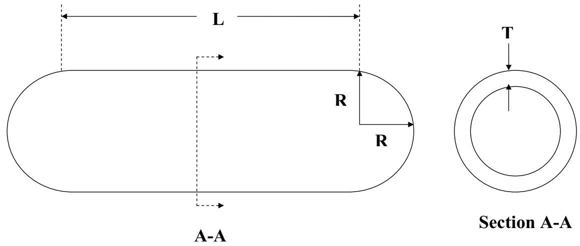


Figure 1. THIN-WALLED PRESSURE VESSEL

| | |
|---|---|
| W | Weight of the pressure vessel ($lbs$) |
| V | Volume ($in^3$) |
| R | Radius ($in$) |
| T | Thickness ($in$) |
| L | Length ($in$) |
| P | Pressure inside the cylinder ($klb$) |
| $S_t$ | Material allowable tensile strength ($klb$) |
| $\rho$ | Density of the material ($lbs/in^3$) |
| $\sigma_{circ}$ | Circumferential stress ($lbs/in^2$) |

Table 2. NOMENCLATURE OF THE PRESSURE VESSEL

$$\textbf{VOL} \quad R(T) = 29.29 + 14.75T - 10.01T^2$$
$$L(T) = 85.45 - 34.45T + 20.10T^2 \tag{20}$$

$$\textbf{WGT} \quad T(R,L) = 2 + 1.75R + 0.2445R^2 \tag{21}$$

The problem constants for these RRS are:

$$P = 3.89\,klb; \qquad S_t = 35.0\,klb; \qquad \rho = 0.283\,lbs/in^3 \tag{22}$$

The intersection of the two RRS surfaces corresponds to the equilibrium point:

$$\begin{aligned} R &= 28.4\,in \\ L &= 87.5\,in \\ T &= 3.09\,in \end{aligned} \tag{23}$$

| **Subsystem VOL** | |
|---|---|
| *Maximize* | $V(R,L) = \frac{4}{3}\pi R^3 + \pi R^2 L$ |
| *Design Variables* | $R$ and $L$ |
| *Stress constraint* | $\sigma_{circ} = \frac{PR}{T} \leq S_t$ |
| *Geometric constraints* | $5T - R \leq 0$ |
| | $R + T - 40 \leq 0$ |
| | $L + 2R + 2T - 150 \leq 0$ |
| *Side constraints* | $0.1 \leq R \leq 36$ |
| | $0.1 \leq L \leq 140$ |

Table 3. MODEL OF SUBSYSTEM VOL

Normalizing the variables to lie between -1 and 1 and subsequently moving the equilibrium point to the origin results in the equations which represent the update equations for the iterative design process:

$$\begin{aligned} R(k+1) &= 0.887\,T(k) - 0.558\,T(k)^2 \\ L(k+1) &= -0.525\,T(k) + 0.287\,T(k)^2 \\ T(k+1) &= 0.739\,R(k) + 0.089\,R(k)^2 \end{aligned} \tag{24}$$

It can be seen that the $R$ and $T$ equations are functions of each other, implying that the update equation for $L$ does not play a role in the convergence and can be ignored for the determination of the domain of attraction.

The final update equations for the analysis are given by

$$R(k+1) = 0.887T(k) - 0.558T^2(k) \tag{25}$$

$$T(k+1) = 0.739R(k) + 0.089R^2(k) \tag{26}$$

| **Subsystem WGT** | |
|---|---|
| *Minimize* | $W(R,T,L) = \rho\left[\frac{4}{3}\pi(R+T)^3\right.$ |
| | $\left.+\pi(R+T)^2L - \left(\frac{4}{3}\pi R^3 + \pi R^2 L\right)\right]$ |
| *Design Variables* | $T$ |
| *Stress constraint* | $\sigma_{circ} = \frac{PR}{T} \leq S_t$ |
| *Geometric constraints* | $5T - R \leq 0$ |
| | $R + T - 40 \leq 0$ |
| | $L + 2R + 2T - 150 \leq 0$ |
| *Side constraints* | $0.5 \leq T \leq 6$ |

Table 4. MODEL OF SUBSYSTEM WGT

Chanron et al. [23] illustrate the stability of the origin (equilibrium point) by exploiting the sum of squares approach using SOSTOOLS [28] to determine a Lyapunov function which illustrates that the origin is locally stable. A contour plot of the expression $V(k+1) - V(k)$ which should be negative definite for stability, was used to illustrate the domain of attraction. The Lyapunov function proposed in [23], found through trial and error is shown in Eq.(27).

$$V = 2.54R^2 + 0.40RT + 3.96T^2, \tag{27}$$

The Zero Detection Algorithm was employed to determine the provable domain of attraction for this Lyapunov function and Figure 2 illustrates the resulting domain of attraction. It can be seen that the ellipse $V = \gamma = 1.24$, is tangential to the contour curve which corresponds to $V(k+1) - V(k)$=0. The square box corresponds to the domain of interest, i.e., the normalized design space defined by the design variable side constraints

It is clear from Figure 2 that there are large regions of the design space which are not included in the domain of attraction. To improve the estimate of the domain of attraction, the expanding-interior approach is studied.

For this example, an arbitrary shape function $p(R,T) = R^2 + T^2$ is used. A sixth order Lyapunov function shown in Figure 3 is found using the expanding interior algorithm. It can be seen that this Lyapunov function proves stability in the entire design space $\{-1.5766 \leq R \leq 0.4234,\ -0.9418 \leq T \leq 1.0582\}$. The convergence of the algorithm to a Lyapunov function depends

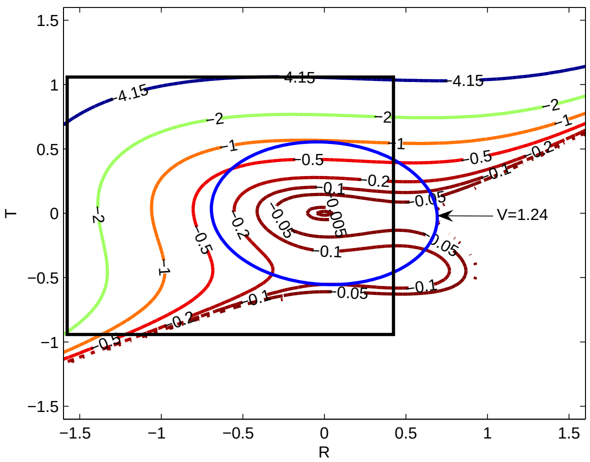


Figure 2. DOMAIN OF ATTRACTION

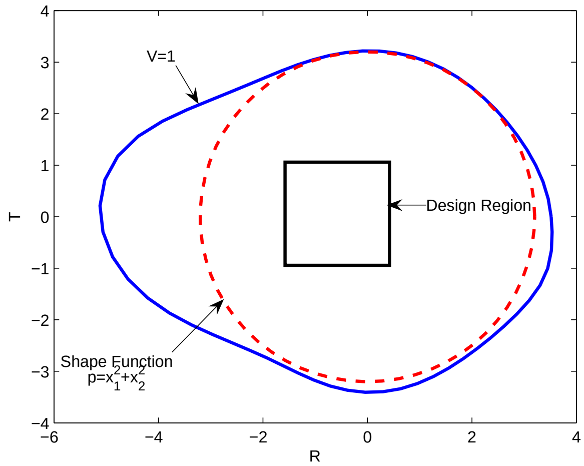


Figure 3. DOMAIN OF ATTRACTION $V \leq 1$ USING EXPANDING INTERIOR ALGORITHM

on the choice of the shape function. That is, a different shape function will result in a different Lyapunov function.

In Figure 4, the new domain of attraction (with $n_v = 6$) is compared to the domain found in [23] (with $n_v = 2$). It is clear that the new domain of attraction is significantly better than the previous one.

The algorithm iteratively searches for a new Lyapunov function starting from an initial Lyapunov function. Thus the choice of the initial Lyapunov function is crucial to the performance of the algorithm. To illustrate that the estimate of the domain of attraction of the equilibrium point is a function of the initial Lyapunov function, we consider the linearized approximation of

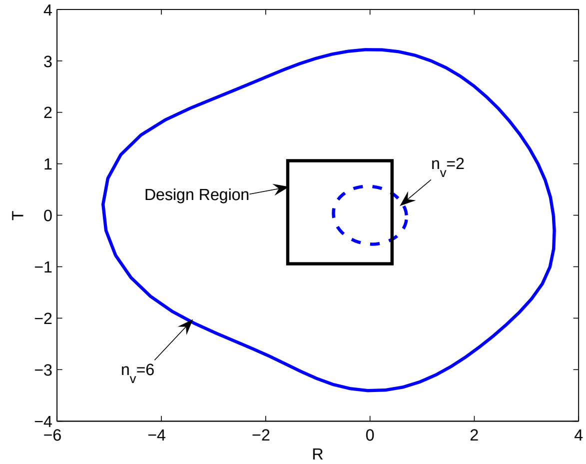


Figure 4. COMPARING THE ESTIMATES OF THE DOMAIN OF ATTRACTION

Eqs. (25) and (26):

$$R(k+1) = 0.887T(k) \tag{28}$$
$$T(k+1) = 0.739R(k). \tag{29}$$

which can be represented in state space form as:

$$\begin{Bmatrix} R \\ T \end{Bmatrix}(k+1) = \underbrace{\begin{bmatrix} 0 & 0.8870 \\ 0.7390 & 0 \end{bmatrix}}_{A} \begin{Bmatrix} R \\ T \end{Bmatrix}(k). \tag{30}$$

If the origin is stable, the Lyapunov Equation:

$$APA' - P + Q = 0 \tag{31}$$

should result in a positive definite matrix $P$ for any positive definite matrix $Q$. Solving Eq. (31) for $Q = I$, the identity matrix, results in the Lyapunov function:

$$V = \{R\ T\}P\begin{Bmatrix} R \\ T \end{Bmatrix} = 3.1329R^2 + 2.7109T^2. \tag{32}$$

The expanding interior algorithm is used to determine a sixth order Lyapunov function using Eq. (32) as the initial Lyapunov function. The domain of attraction for the resulting sixth order

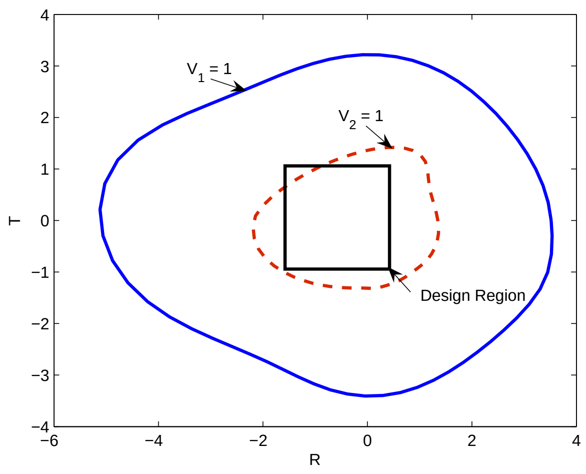


Figure 5. COMPARING THE DOMAIN OF ATTRACTION FOR DIFFERENT INITIAL LYAPUNOV FUNCTIONS

Lyapunov function ($V_2$) is illustrated by the dashed line in Figure 5 and is compared to the Lyapunov function ($V_1$) generated by initializing the algorithm with Eq. (27).

It is clear from Figure 5 that the initial Lyapunov function plays an important role in the determination of the domain of attraction of the equilibrium point. But, regardless, the provable domain of attraction using the two approaches is guaranteed to always be larger than the domains developed in [23].

## 5 CONCLUSIONS AND FUTURE WORK

Two approaches which exploit the Positivstellensatz theorem to determine the domain of attraction for continuous time systems have been modified to cater to discrete time systems. These approaches have been used to determine the stability region of equilibrium points of a decentralized design system. The benchmark pressure vessel design problem has been studied and the benefit of using the expanding interior approach over the zero detection approach has been demonstrated. The proposed approach can be automated to generate the domain of attraction for generic discrete time systems. The expanding interior approach is an iterative algorithm which is not guaranteed to converge to a global optimum. This requires initiating the algorithm with different Lyapunov functions. The shape function is also instrumental in the effectiveness of the algorithm. But the two approaches presented, significantly improve the estimate of the domain of attraction. By generating improved domains of attraction, the convergence to a stable equilibrium solution of decentralized design problems can be guaranteed across a much wider range, and many times across the entire design space. This implies, that regardless of where the process starts, and which subsystem starts the process, it will always converge to a stable solution, acceptable to all the subsystems.

Also at the core of distributed processes, is the issue of distributed rationality where individual subproblems each operate in a rational way which may not all align. That is, one subproblem decision maker may make decisions that are rational for them but completely irrational for the other subproblems. This creates scenarios of conflict and compromise that typically lead to suboptimal solutions. These solutions, more commonly known as Nash solutions are equilibrium points, but are not guaranteed to be optimal for any subproblem involved. The practical impact of this is noted in a study on the huge inefficiencies of the distributed decisions that are made in the design and fabrication of a building when designers, architects, engineers, developers and builders each make decision that serve their own interests [29]. Therefore, in future work, we will focus on developing convergence characteristics, including domains of attractions that lead the process to collective optimal solutions (also known as Pareto solutions).

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