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CHARACTERISTICS OF DEFLECTION-LIMITED TIME-OPTIMAL CONTROL OF THE BENCHMARK PROBLEM

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ABSTRACT
The focus of this paper is on the design of time optimal control profiles for flexible structures subject to deflection constraints. The benchmark floating oscillator is used to illustrate the variation in the structure of the control profile as a function of permissible deflection. The transition from a 5 switch bang-bang to a 7 switch bang-bang to finally, a 7 switch profile which includes non-saturating intervals, is demonstrated. The loss of anti-symmetry of the control profile and the transition of the structures of the deflection constrained time-optimal control profiles for damped systems is also presented.
Keywords: Time-Optimal, Deflection constraint, Benchmark Problem.

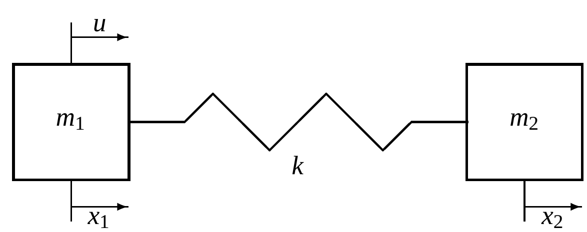

Figure 1. Undamped Floating Oscillator

1 INTRODUCTION
Vibration control of slewing flexible structures has been a subject of research interest in both the aerospace and robotics community [1]- [2]. These studies encompass a number of lightweight flexible structures; such applications include large spacecraft and space structures [3], robotic arms [4], gantry cranes [5], hard disk drives [6], etc. In 1957, Smith [1] proposed a wave cancelation technique, termed "Posicast", to drive a system with one resonant mode to its final position in finite time. Singer and Seering [7] arrived at the same results as Smith with an input shaping approach. In addition, they proposed a technique for making input shaping commands insensitive to errors in the model parameters which involved forcing the system's residual energy, and derivative with respect to the natural frequency or damping, to zero. Singh and Vadali [2] derived the same results as Singer and Seering [7] with the design of a time delay prefilter which provided zeros to the system so as to cancel the poles. Liu and Wie [8] proposed an approach for desensitizing time-optimal control profiles to model uncertainties, which involved decoupling the equations of motion into rigid and flexible body modes. Robustness is achieved by forcing the partial derivative of the decoupled states with respect to natural frequency to zero at the final maneuver time. Singh and Vadali [9] improved the robustness of time-optimal control strategies by applying the pole cancelation technique and requiring the control sequence to satisfy robustness constraints.

The aforementioned techniques essentially concentrate on eliminating residual vibration and increasing the robustness of input shaping and minimum time controllers. None, however, address the problem of limiting the large deflection amplitudes normally associated with them. Take for example the system illustrated in Fig.1, it may be necessary to move the structure a finite distance in minimum time while limiting the maximum extension/contraction that occurs in the spring. Singhose et al. [10]

proposed the first approach for developing deflection-limited input shaping commands. In their paper, a technique for applying deflection limits at instances where local extrema points occur in the transient deflection is presented. However, the expressions are only derived for control impulse amplitudes of ± 1 warranting the development of bang-off-bang control sequences which are not time optimal. Robertson and Singhose [11]- [12] arrived at the same results as the aforementioned with a discrete approach. Later they presented an approach for developing closed form expressions to the deflection-limiting commands [11]. Their technique incorporated magnitude restricted coasting periods in the preshaped profile. One year later, they also developed a robust approach which involved an extension in the number of profile switch times [12].

The aforementioned papers constrain the control profile to be either bang-off-bang [10], or non-saturating bang-off-bang [11]- [12] which are control profiles which do not correspond to the time-optimal solutions. In Section 2 the optimal control problem is described. The following section presents the parameterizations of the control profiles. The numerical results section presents the time-optimal solutions for specific deflections. Charts which illustrate the variation of the switch times as a function of the permitted deflection are presented for both the damped and undamped benchmark problem.

2 MATHEMATICAL FORMULATION

For time optimal control of flexible structures, it is desirable to establish a control strategy that results in quiescent terminal states in the shortest time possible. To guarantee the control input is obtainable in real situations, constraints are placed on the magnitude of the available control input. Furthermore these structures contain permissible deflection limits for the flexible appendages which ensure they are not driven to the point of yielding or failure. To avoid this, constraints on the amount of transient state deformation the system experiences during the maneuver are included.

The traditional time-optimal control problem consist of determining the control $u(t)$, which drives the system states x, governed by $\dot{x}(t) = Ax(t) + Bu(t)$, from their specified initial state x_0 to their desired final state x_f while minimizing the performance index

$$\min J = \int_0^{t_f} dt \tag{1}$$

To guarantee the input magnitude is physically obtainable and the actuators are capable of performing the desired maneuver it must also satisfy the constraint:

$$u_{min} \le u(t) \le u_{max} \tag{2}$$

To ensure the states are not driven beyond the systems permissible level of deflection it must additionally satisfy

$$\Delta x_{min} \le x_1(t) - x_2(t) \le \Delta x_{max} \tag{3}$$

For simplicity the control limits are assumed to be symmetric. In addition, for generality purposes, the control input is constrained to $-1 \le u(t) \le 1$, and the state deflection is constrained to $-\delta \le x_1(t) - x_2(t) \le \delta$, where δ is the specified deflection limit. The resulting problem statement for deflection-limited time-optimal control is

$$\min J = \int_0^{t_f} dt \tag{4a}$$

$$\text{subject to} \quad \dot{x}(t) = Ax(t) + Bu(t) \tag{4b}$$

$$x(t_0) = x_0 \quad x(t_f) = x_f \tag{4c}$$

$$-1 \le u(t) \le 1 \ \ \forall t \tag{4d}$$

$$-\delta \le x_1(t) - x_2(t) \le \delta \ \ \forall t \tag{4e}$$

3 PARAMETERIZATION OF CONTROL PROFILE

Deformation limiting control strategies are a modification of conventional control strategies [13]. The time optimal control strategy for an undamped system is a bang-bang profile which is antisymmetric about the mid maneuvering point. In addition the time optimal profile is fully saturated throughout the entire maneuver, ensuring the final maneuver time is minimized. However, if throughout the maneuver the system experiences excessive deformation then a decrease in control input is needed [13]. Intuitively when the deformation constraint is reached the states of the system must approach the same velocity. As long as the constraint remains active the velocities must remain equal. Therefore the ideal profile would be a bang-bang to initialize the velocity states, then a period of coasting where the velocities remain equal. As with the traditional time optimal control profile we still require the same antisymmetric characteristic to guarantee the states reach the desired set point. Thus the proposed control profile is illustrated in Fig. 2. It is characterized by seven switch times and has a mid maneuver point at T_4. The first half of the maneuver accelerates the system while maintaining control over deformation and vibration, the second half decelerates the system in the same manner. This control strategy can be characterized by a set of Heaviside function shown in Eq. (5).

$$\begin{aligned} u(t) = 1 &- 2\mathcal{H}(t-T_1) + (1+\Delta)\mathcal{H}(t-T_2) \\ &- (1+\Delta)\mathcal{H}(t-T_3) + 2\mathcal{H}(t-T_4) - (1+\Delta)\mathcal{H}(t-T_5) \\ &+ (1+\Delta)\mathcal{H}(t-T_6) - 2\mathcal{H}(t-T_7) + \mathcal{H}(t-T_8) \end{aligned} \tag{5}$$

Here Δ represents the input magnitude necessary for the coasting period of the two masses. It was previously stated

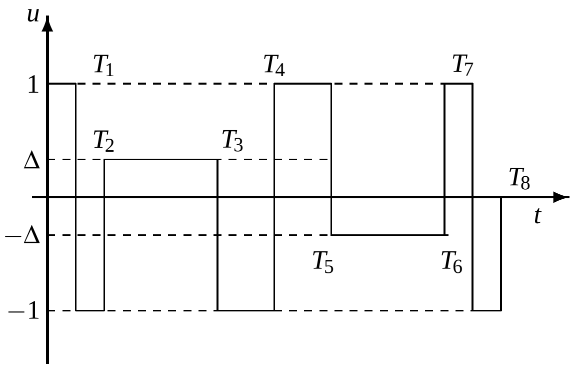

Figure 2. Deformation Limited Time Optimal Control

that during this maneuvering period the velocities of each mass needed to remain equal. While the input magnitude remains constant, $u(t) = \Delta$, in order for the velocity states to remain equal, $\dot{x}_1(t) = \dot{x}_2(t)$, each mass must accelerate at the same constant rate. Thus the acceleration states must be equal and constant, represented by

$$\ddot{x}_1(t) = \ddot{x}_2(t) = const \tag{6}$$

Substituting the equations of motion into Eq. (6) and letting the control input assume the value $u(t) = \Delta$ leads to

$$\frac{\Delta}{m_1} - \frac{k(x_1(t) - x_2(t))}{m_1} = \frac{k(x_1(t) - x_2(t))}{m_2} \tag{7}$$

Solving for Eq. (7) in terms of Δ gives

$$\Delta = \frac{(m_1 + m_2)k}{m_2}(x_1(t) - x_2(t)) \tag{8}$$

Recall that during this maneuver the position states are coasting along the constraint bound. That is the displacement between the two masses is being held at the maximum or minimum allowable deformation, or

$$[x_1(t) - x_2(t)]|_{u(t)=\Delta} = \delta \tag{9}$$

Substituting Eq. (9) into Eq. (8) leads to

$$\Delta = \frac{(m_1 + m_2)k}{m_2}\delta \tag{10}$$

Eq. (10) represents the magnitude of control input needed to preform the coasting periods of the maneuver. This expression is further reduced when the parameters assume a value of unity, $m_1 = m_2 = k = 1$

$$\Delta = 2\delta \tag{11}$$

Substituting this expression back into Eq. (5) gives

$$\begin{aligned} u(t) = {} & 1 - 2\mathcal{H}(t - T_1) + (1 + 2\delta)\mathcal{H}(t - T_2) - (1 + 2\delta)\mathcal{H}(t - T_3) \\ & + 2\mathcal{H}(t - T_4) - (1 + 2\delta)\mathcal{H}(t - T_5) + \\ & (1 + 2\delta)\mathcal{H}(t - T_6) - 2\mathcal{H}(t - T_7) + \mathcal{H}(t - T_8) \end{aligned} \tag{12}$$

Eq. (12) represent the fully parameterized control strategy as a function of switch times T_i and permissible deformation δ.

3.1 Deformation Constraint

The control strategy along with all the constraint equations are parameterized in terms of switch times. Therefore it would be desirable to derive a closed form expression for the systems deformation in terms of the switch times.

The total deformation in the Laplace domain may be represented as

$$X_1(s) - X_2(s) = \frac{1}{s^2 + \omega^2}U(s) \tag{13}$$

By taking the inverse Laplace transformation of the Eq. (13) the deformation expression may be derived as a function of time, resulting in

$$\begin{aligned} \omega^2(x_1(t) - x_2(t)) = {} & [1 - \mathcal{C}(\omega t) - 2(1 - \mathcal{C}(\omega(t - T_1)))\mathcal{H}(t - T_1) \\ & + (1 + 2\delta)(1 - \mathcal{C}(\omega(t - T_2)))\mathcal{H}(t - T_2) \\ & - (1 + 2\delta)(1 - \mathcal{C}(\omega(t - T_3)))\mathcal{H}(t - T_3) \\ & + 2(1 - \mathcal{C}(\omega(t - T_4)))\mathcal{H}(t - T_4) \\ & - (1 + 2\delta)(1 - \mathcal{C}(\omega(t - T_5)))\mathcal{H}(t - T_5) \\ & + (1 + 2\delta)(1 - \mathcal{C}(\omega(t - T_6)))\mathcal{H}(t - T_6) \\ & - 2(1 - \mathcal{C}(\omega(t - T_7)))\mathcal{H}(t - T_7) \\ & + (1 - \mathcal{C}(\omega(t - T_8)))\mathcal{H}(t - T_8)] \end{aligned} \tag{14}$$

where $\mathcal{C}(.) = cos(.)$.

Therefore the state deformation that occurs between any two switch times, may be represented by

$$x_1(t) - x_2(t) = \frac{1}{\omega^2}[1 - \mathcal{C}(\omega t) + \sum_{j=1}^{i} A_j(1 - \mathcal{C}(\omega(t - T_j)))] \tag{15}$$

$\forall t \in [T_i \; T_{i+1}]$, where n represents the total number of profile switch times and A_i denotes the input magnitude change at each corresponding switch time T_i.

Note that Eq. (15) is only valid for time t existing in the interval $t \in [T_i \; T_{i+1}]$. Therefore Eq. (15) may be used to compute a series of deformation expressions between each switch time interval in the control profile. To satisfy the inequality constraint the deformation values that are in strictest violation of the constraint bound have to be considered. Therefore the maximum deformation occurring in each switch interval needs to be determined. We begin by taking the time derivative of Eq. (15) which is given by

$$\frac{d}{dt}[x_1(t) - x_2(t)] = \frac{1}{\omega}[\sin(\omega t) + \sum_{j=1}^{i} A_j(\sin(\omega(t - T_j)))] \quad (16)$$

$\forall t \in [T_i \; T_{i+1}]$. By equating Eq. (16) to zero and solving for t, the time at which a local maxima or minima occurs may be represented by

$$t_m = \frac{1}{\omega}\arctan\{\frac{\sum_{j=1}^{i} A_j \sin(\omega T_j)}{1 + \sum_{j=1}^{i} A_j \sin(\omega T_j)}\} + \frac{\pi k}{\omega} \quad (17)$$

for $k = 0, 1, 2, \ldots$. Note that Eq. (17) is nonlinear and therefore can generate multiple solution in intervals of $\frac{\pi}{\omega}$. Therefore if one or more solutions to Eq. (17) lie within the interval, $t_m \in [T_i \; T_{i+1}]$, then a local extreme point exist. The deformation value at these times may then be determined by evaluating Eq. (15) at t_m. However, there are certain cases when no local extreme points exist in the interval of interest. In these instances the maximum deformation can only occur at the interval boundaries. Therefore the maximum deformation that occurs in any switch time interval may be represented by

$$\Delta x_{max}(i) = \max |\Delta x(T_i) \;\; \Delta x(t_m) \;\; \Delta x(T_{i+1})| \quad (18)$$

$\forall t_m \in [T_i \; T_{i+1}]$, where the function Δx in Eq. (18) is a shortened notation for the deformation expression, $x_1(t) - x_2(t)$.

Finally, the maximum deformation value that occurs across the entire time span of the control strategy may be determined by taking the maximum of Δx_{max} which occurs in each switch interval. The final state deformation constraint may be represented in the form

$$\max[\Delta x_{max}(i)] \leq \delta \quad for \; i = 0 \ldots n \quad (19)$$

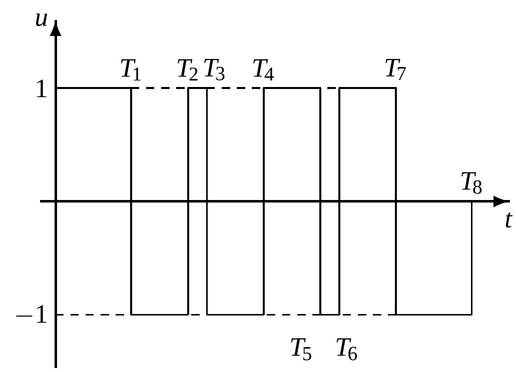

Figure 3. Intermediate Deformation Limited Control

3.2 Intermediate Control Profile

When the allowable level of transient deformation is large, the parametrization of the deformation limiting control strategy simply collapses into the conventional time optimal solution. However, as derived in Eq. (11), the deformation controlled maneuvering requires $u(t) = 2\delta$. Therefore there exist certain levels of permissable deformation for which the control input will violate its saturation constraint $-1 \leq u(t) \leq 1$. For this reason we require the parametrization of an intermediate profile to make the transition between the time optimal solution and the unsaturated deformation limiting control in Fig. 2. Figure 3 illustrates the proposed intermediate control profile which can be represented in the Laplace domain as

$$U(s) = \frac{1}{s}[1 - 2e^{sT_1} + 2e^{sT_2} - 2e^{sT_3} + 2e^{sT_4} - 2e^{sT_5} + 2e^{sT_6} - 2e^{sT_7} + e^{sT_8}] \quad (20)$$

For each of the parametrization, a parameter optimization problem can be derived with constraints to satisfy the boundary conditions and the deflection constraints. Any gradient based optimization algorithms can be used to solve for the switch time.

4 NUMERICAL SIMULATIONS

In this section, the proposed techniques are illustrated on the undamped floating oscillator problem (Fig.1) undergoing a rest-to-rest maneuver whose equations of motion are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \quad (21)$$

with the boundary conditions

$$\underline{x}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \underline{\dot{x}}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \underline{x}(t_f) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \underline{\dot{x}}(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22}$$

4.1 Non-restrictive Deformation

The first simulation was performed using a large enough δ that the deformation constraint remained inactive. Thereby the solution is based solely on the conditions for time optimality. The control profile was found to be

$$\begin{aligned} u(t) = 1 &- 2\mathcal{H}(t-1.0027) + 2\mathcal{H}(t-2.1089) \\ &- 2\mathcal{H}(t-3.2152) + 2\mathcal{H}(t-4.2179) \end{aligned} \tag{23}$$

The control strategy as well as the system response are illustrated in Fig. 4. This is the traditional time optimal solution for this problem.

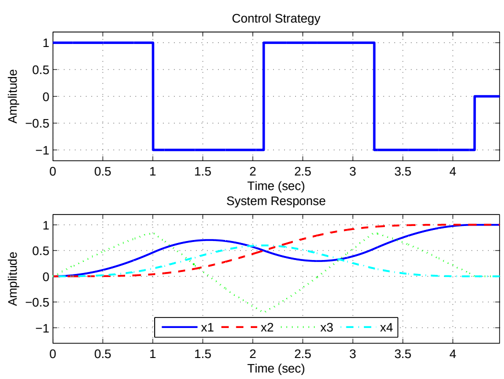

Figure 4. Control Profile for Non-restricting δ

4.2 Intermediate Control Profile

In this example, the control profile was computed for an allowable deformation of $\delta = 0.50$, thus resulting in an intermediate profile as discussed in Section 3.2. The control profile,

$$\begin{aligned} u(t) = 1 &- 2\mathcal{H}(t-0.9320) + 2\mathcal{H}(t-1.2894) - 2\mathcal{H}(t-1.3954) \\ &+ 2\mathcal{H}(t-2.1359) - 2\mathcal{H}(t-2.8764) + 2\mathcal{H}(t-2.8924) \\ &- 2\mathcal{H}(t-3.3397) + \mathcal{H}(t-4.2717) \end{aligned} \tag{24}$$

remains fully saturated throughout the entire maneuver and is shown in Fig. 5 along with the evolution of the system states.

The state deformation for this simulation is illustrated in Fig. 6. The dashed lines represent the deformation limits $\pm\delta$. Note that the deformation is saturated for all time during the coasting maneuver.

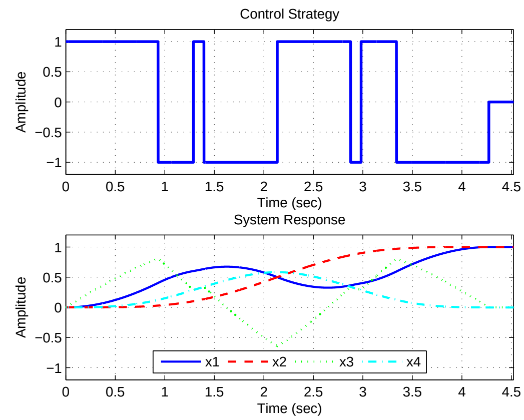

Figure 5. Deformation Limited Control for $\delta = 0.50$

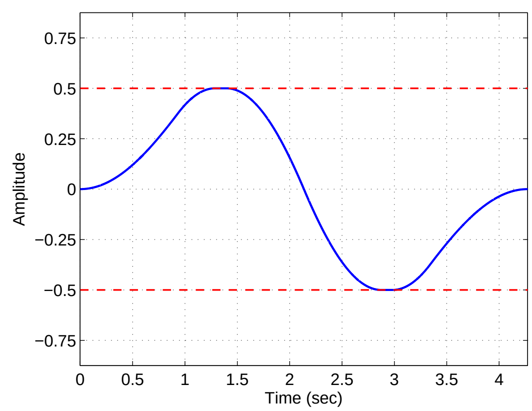

Figure 6. State Deformation for $\delta = 0.50$

4.3 Deformation limited Control Profile

The allowable deformation in this example was selected to be $\delta = 0.25$, hence the input limited period, where $u(t) = 2\delta$, is no longer saturated. The control strategy was found to be

$$\begin{aligned} u(t) = 1 &- 2\mathcal{H}(t-0.575) + 1.5\mathcal{H}(t-0.932) - 1.5\mathcal{H}(t-1.908) \\ &+ 2\mathcal{H}(t-2.503) - 1.5\mathcal{H}(t-3.098) + 1.5\mathcal{H}(t-4.074) \\ &- 2\mathcal{H}(t-4.432) + \mathcal{H}(t-5.006) \end{aligned} \tag{25}$$

Shown in Fig. 7 is an illustration of this control strategy along with the system response. In Section 3 the presumption of the velocity states remaining equal and the acceleration remaining constant becomes more evident in this figure. Note at the beginning of each coasting period the velocities, x_3 and x_4, merge together and track along a straight path; they do not separate until the next switch time.

Also shown in Fig. 8 is the transient deformation for this control strategy. Here the limitations, $\pm\delta$, that are reached during

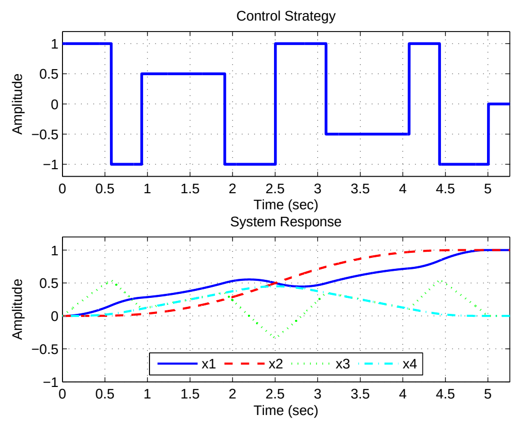

Figure 7. Deformation Limited Control for $\delta = 0.25$

the coasting maneuver are better exposed.

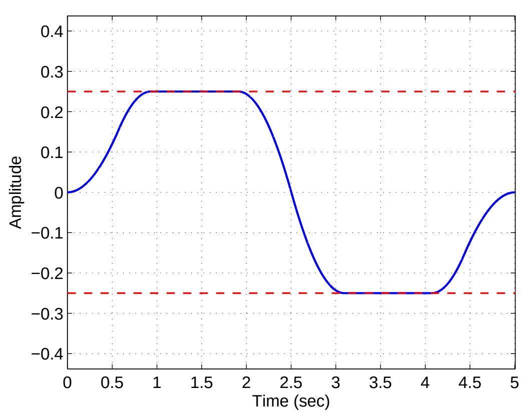

Figure 8. State Deformation for $\delta = 0.25$

4.4 Switch Time Trajectories

Figure 9 illustrates the transition in control structure based on the profile switch times for varying magnitudes of allowable deformation. The figure clearly demonstrates that as the permissible deformation tends toward zero, $\delta \longrightarrow 0$, the maneuver time tends towards infinity, $t_f \longrightarrow \infty$. In addition it shows the transition into the intermediate profile when $\delta = 0.50$. As the allowable deformation approaches $\delta \longrightarrow 0.5477$, switch times T_2, T_3, T_5, and T_6, which characterize the coasting periods for deformation control, collapse which leads to the time optimal solution. Intuitively as $\delta \longrightarrow \infty$ the unconstrained time optimal solution remains.

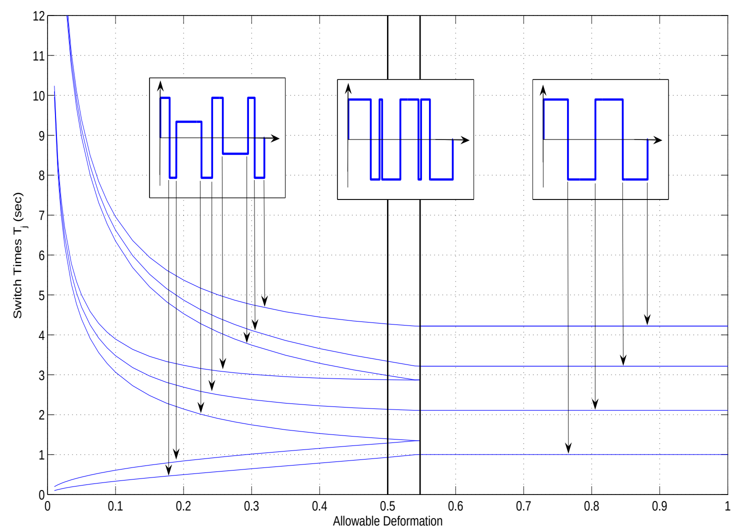

Figure 9. Trajectories of the Switching Times

4.5 Switch Time Trajectories (Damped System)

The transition in control structure with respect to the permissible level of deformation for the benchmark floating oscillator with a damping ratio of 0.25, is presented in Fig. 10. Similar to the results in the section on the undamped system, as the allowable deformation tends toward zero, $\delta \longrightarrow 0$, the final maneuver time tends towards infinity, $t_f \longrightarrow \infty$. However the most apparent variation between the undamped and damped case is illustrated by the antisymmetric collapse in the coasting period switch times, characterized by T_2, T_3, T_5, and T_6. As the allowable deformation approaches $\delta \longrightarrow 0.4938$ the first two switch times, characterized by T_2, and T_3, collapse, which is just prior to the deformation limit of $\delta = 0.5$. This further clarifies why only five switch times are present in the intermediate profile. Moreover, once the permissible deformation surpasses 0.5 the second coasting period, characterized by T_5, and T_6, becomes fully saturated. As the limit tend towards $\delta \longrightarrow 0.5438$ the second pair of switch times collapse which leads to the time optimal solution. Clearly as $\delta \longrightarrow \infty$ the unconstrained time optimal solution remains.

5 CONCLUSIONS

The variation in the structure of deflection limited time-optimal control profile for the benchmark floating oscillator is presented. The control profile is parameterized in terms of the permissible deflection leading to posing the problem as a parameter optimization problem. Results illustrate that the deflection limited time-optimal control profile varies from a 5 switch bang-bang form which correspond to large permissible deflection to a 7 switch bang-bang form which is referred to as an intermedi-

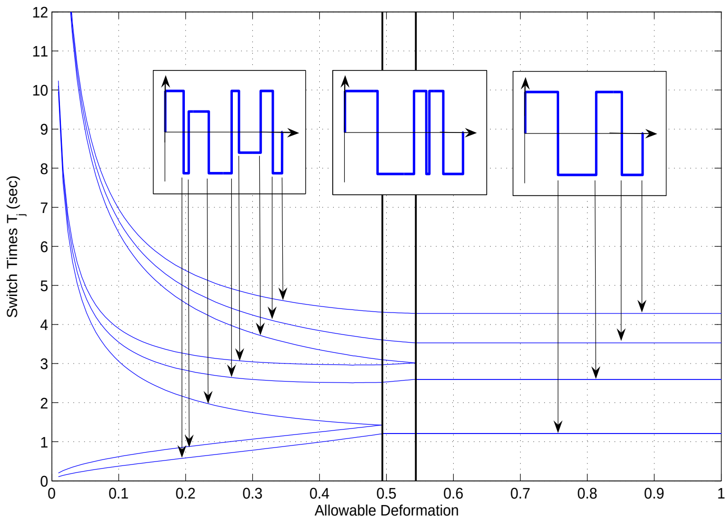

Figure 10. Trajectories of the Switching Times ($\zeta = 0.25$)

ate control profile to a 7 switch non-saturating switching control profile which corresponds to small permissible deflection.

REFERENCES

[1] O. J. M. Smith. Posicast control of damped oscillatory systems. In *Proceedings of the IRE*, pages 1249–1255, September 1957.

[2] T. Singh and S. R. Vadali. Robust time-delay control. *ASME Journal of Dynamic Systems, Measurement and Control*, 115(2(A)):303–306, 1993.

[3] J. L. Junkins, Z. H. Rahman, and H. Bang. *Near-Minimum Time Maneuvers of Flexible Vehicles: A Lyapunov Control Law Design Method*. AIAA Publication Series. AIAA, Washington, DC, mechanics and control of large flexible structures edition, 1990.

[4] W. L. Ballhaus, S. M. Rock, and A. E. Bryson. Optimal control of a two-link flexible robotic manipulator using time-varying controller gains. *American Astronautics Society*, Paper 92-055, 1992.

[5] W. E. Singhose, L. J. Porter, and W. P. Seering. Input shaped control of a planar gantry crane with hoisting. In *1997 American Control Conference*, Albuquerque, NM.

[6] S. P. Bhat and D. K. Miu. Minimum power and minimum jerk control and its application in computer disk drives. *IEEE Transactions on Magnetics*, 27(6):4471–4475, 1991.

[7] N. C. Singer and W. P. Seering. Preshaping command inputs to reduce system vibrations. *ASME Journal of Dynamic Systems, Measurement and Control*, 115:76–82, 1990.

[8] Q. Liu and B. Wie. Robust time-optimal control of uncertain flexible spacecraft. *AIAA Journal of Guidance, Control and Dynamics*, 15(3):597–604, 1992.

[9] T. Singh and S. R. Vadali. Robust time-optimal control: Frequency domain approach. *AIAA Journal of Guidance, Control and Dynamics*, 17(2):346–353, 1994.

[10] A. K. Banerjee W. E. Singhose and W. P. Seering. Slewing flexible spacecraft with deflection-limiting input shaping. *Journal of Guidance, Control, and Navigation*, 20(2):291–297, 1997.

[11] M. J. Robertson and W. E. Singhose. Closed-form deflection-limiting commands. *American Control Conference*, 11(3):2104–2109, 2005.

[12] M. J. Robertson and W. E. Singhose. Robust analytic deflection-limiting commands. *American Control Conference*, 11(2):363–368, 2006.

[13] K. Hirohisa and W. Singhose. Adaptive deflection limiting control for slewing flexible space structures. *Journal of Guidance, Control, and Navigation*, 30(1):61–67, 2007.