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Desensitized Minimum Power/Jerk Control Profiles for Rest-to-Rest Maneuvers

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Abstract

The focus of this paper is on the development of weighted minimum power/jerk control profiles for the rest-to-rest maneuver of a flexible structure. To account for modeling uncertainties, equations are derived which represent the sensitivity of the system states to model parameters. The original state-space model of the flexible structure is augmented with the sensitivity state equations with the constraint that the sensitivity state variables are forced to zero at the end of the maneuver. This requirement attenuates the residual vibration at the end of the maneuver caused by errors in system parameters. A systematic procedure for the design of the controller is developed by representing the linear-time-invariant system in its Jordan form. This decouples the modes of the system permitting us to address smaller order dynamical systems. The proposed technique is illustrated via a benchmark floating oscillator problem.

1 Introduction

The control of the benchmark two-mass/spring/damper system undergoing a rest-to-rest maneuver is considered in this paper. This problem, representative of many flexible structures, has one flexible mode and one rigid body mode. A fairly comprehensive treatment of this family of problems has been presented by Junkins and Turner [1]. In previous research on this topic, time optimal control profiles have been derived by Singh et al. [2], Ben-Asher et al. [3], and Hablani [4]. Desensitizing the control profiles to modeling errors has been addressed by Liu and Wei [5] and Singh and Vadali [6]. Closed-form solutions have been obtained for the optimal control of the rest-to-rest maneuver using minimum power and minimum jerk cost functions by Bhat and Miu [7]. Recently, it has been of interest to develop optimal solutions using a weighted cost function, such as the weighted fuel/time optimal control considered by Singh [8]. Here, the closed-form solution for the optimal control of the rest-to-rest maneuver using a weighted minimum power/jerk cost function is of interest. In the weighted cost function considered here, the user can select the relative importance of power (or equivalently control effort) to jerk (or equivalently the rate of change of control effort).

The solution for the control profile obtained for linear-time-invariant systems, like the system considered in this paper, often assume known constant system parameters. This assumption is not valid in actual physical systems since the system parameters cannot be determined exactly. With this in mind, it is the goal of the researchers to obtain a solution which is robust to errors in system parameters (e.g. - damping ratio, natural frequency). To do this, sensitivity equations are derived and added to the state space equations before transforming them into Jordan canonical form. It will be shown that with the addition of these equations, which force the sensitivity state variables to zero at the end of the maneuver, there is a reduction in residual vibration due to errors in system parameters.

The paper begins with the problem formulation in Section 2. Section 3 gives a numerical example with results presented. The topic of sensitivity equation formulation is considered in Section 4. Section 5 gives the same numerical example considered in Section 3, this time with the addition of sensitivity equations. A comparison between the robust and nonrobust solutions is also drawn in this section. Finally, the paper is concluded with a summary of the obtained results in Section 6.

2 Problem Formulation

The weighted minimum power/jerk cost function

$$min\frac{1}{2}\int_0^T \left(\zeta^2 u^2 + \left(\frac{du}{dt}\right)^2\right) d\tau \qquad (1)$$

is considered, subject to the constraint

$$M\ddot{x} + \xi\dot{x} + Kx = Pu \qquad (2)$$

where M is the mass matrix, ξ the damping matrix, and K is the stiffness matrix. P is the control influence vector, and u and x are the scalar control input and state vector, respectively. In Equation 1,

$\zeta = \ln(\alpha)$ ($\zeta \epsilon [0,\infty]$) and T = specified final time, where ζ is the scalar weighting parameter which is a function of α. The scalar α is the parameter to be varied, thus having the effect of varying ζ. The reason that $\zeta = \ln(\alpha)$ is used has to do with simplification of the general form of the solution, which would otherwise contain hyperbolic sinusoidal functions. The equations of motion for this system can be represented in Jordan canonical form as

$$\dot{z} = Jz + bu \tag{3}$$

$$y = C^{*}z + Du \tag{4}$$

The solution of Equation 3 is given as

$$e^{-Jt_2}z(t_2) - e^{-Jt_1}z(t_1) = \int_{t_1}^{t_2} e^{-J\tau}bu(\tau)d\tau. \tag{5}$$

It will be shown that with a parameterized control u, a closed-form solution for the control profile can be obtained using Equation 5. The control u will contain the parameters λ_i where $i = 1, 2, 3, ...n$. Here n is the number of parameters in the control profile, which in a general case will depend on the size of the system and the cost function which is minimized. For the system considered here with one rigid body mode, using the proposed minimum power/jerk cost function yields $n = 4 + 2p$, where the scalar p is the number of flexible modes of the system.

In order for the chosen cost function (Equation 1) to be minimized, the following performance criteria must be minimized

$$I = \frac{1}{2}\int_0^T \left(\zeta^2u^2 + \left(\frac{du}{dt}\right)^2\right)d\tau + \lambda^{\mathrm{T}}\left[e^{-JT}z(T) - \int_0^T e^{-J\tau}bu(\tau)d\tau\right]. \tag{6}$$

Equation 6 is derived by assuming the maneuver time to be T, the initial time and initial conditions to be zero, and by augmenting the cost function with Equation 5. By taking the first variation of this equation and setting it equal to zero, the u which minimizes Equation 1 can be obtained. The first variation is expressed as

$$\delta I = \int_0^T \left[\zeta^2u - \frac{d^2u}{dt^2} - \lambda^{\mathrm{T}}e^{-J\tau}b\right]\delta u d\tau + \left[\frac{du}{dt}\delta u\right]_0^T. \tag{7}$$

In order for this equation to be equal to zero for all δu, the quantities inside the brackets must be equal to zero. This requirement results in a differential equation in u which is

$$\frac{d^2u}{dt^2} - \zeta^2u = -\lambda^{\mathrm{T}}e^{-Jt}b. \tag{8}$$

By taking the Laplace transform of this equation, the solution for the control profile u can be obtained. An additional requirement which must be satisfied, from Equation 7, is that the time derivative of the control must be equal to zero at initial and final time. There is no requirement that the control u be forced to zero at initial and final time, as is done in the minimum jerk solution obtained in [7]. In that paper, the control is forced to zero at initial and final time in order to make the control practical to input on a real physical system, though the cost function is not minimized. Thus, the minimum jerk solution obtained in [7] is suboptimal. Here the optimal solution will be considered, and it will be shown that as ζ goes to zero, the solution converges to the minimum jerk solution, and conversely, the minimum power solution is obtained as ζ goes to infinity.

A general closed-form solution for the weighted minimum power/jerk control can be found by solving Equation 8 with the necessary condition for optimality that the derivative of the control at initial and final time is set equal to zero. The general closed-form solution for a system with one rigid body mode and p flexible modes is given as

$$u(t) = \lambda_1 + \lambda_2 t + \lambda_3\alpha^t + \lambda_4\alpha^{-t} + \sum_{i=1}^{p}\left(\lambda_{(3+2i)}e^{-a_it}\sin(b_it) + \lambda_{(4+2i)}e^{-a_it}\cos(b_it)\right) \tag{9}$$

where a_i is the real part of the i^{th} complex conjugate pole, and b_i is the imaginary part of the i^{th} complex conjugate pole of the system. The parameters (λ_i) in Equation 9 are found by simultaneously solving Equation 5 and the boundary conditions from Equation 7. This procedure will be demonstrated via a numerical example in the next section.

3 Numerical Example 1

The benchmark two-mass/spring/damper problem will now be considered. Figure 1 shows the system to be considered, with the two masses m_0 and m_1, the spring constant k, and a viscous damper c. In the figure, x_0 and x_1 are the displacement of the first and second mass, respectively. The input force is denoted as u, and the output as y.

The system equations are given in state space form as

$$\dot{x} = Ax + Bu \tag{10}$$
$$y = Cx + Du \tag{11}$$

where

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \dot{x_0} \\ \dot{x_1} \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k}{m_0} & \frac{k}{m_0} & \frac{-c}{m_0} & \frac{c}{m_0} \\ \frac{k}{m_1} & \frac{-k}{m_1} & \frac{c}{m_1} & \frac{-c}{m_1} \end{bmatrix} \tag{12}$$

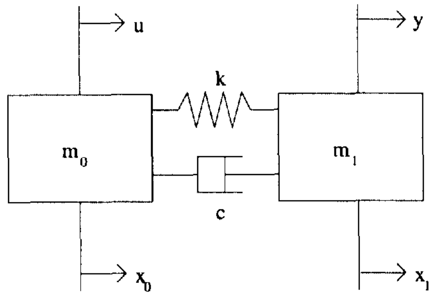

Figure 1: Two-mass/spring/damper system.

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_0} \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, D = 0. \quad (13)$$

When the state space equations are converted into Jordan canonical form, Equation 10 can be rewritten as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}}_{Vx} + \underbrace{\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{bmatrix}}_{VB=b} u \quad (14)$$

where J is the Jordan canonical form of the A matrix, V is the transformation matrix, and p_1 and p_2 are the complex conjugate poles of the flexible mode.

The rest-to-rest maneuver of the two-mass/spring /damper benchmark problem is considered here. For simplicity, the following parameter values of $m_0 = m_1 = 1$, $k = 1/2$, $c = 0.25$, $t_1 = 0$ (initial time), $t_2 = 4\pi$ (final time), and $\alpha = 10$ ($\zeta = \ln(10)$) will be used. The input is on m_0 and the output is the position of m_1. The initial positions of m_0 and m_1 are both zero and the final positions are chosen (arbitrarily) to be one. Using the parameterized closed-form solution of u (Equation 9), the solution is found by rewriting Equation 5 as

$$\begin{bmatrix} e^{-J(4\pi)} z(4\pi) \\ \cdots\cdots\cdots\cdots \\ 0 \\ 0 \end{bmatrix} = S\lambda \quad (15)$$

where S is given as

$$S = \text{jacobian} \begin{bmatrix} \int_{t_1}^{t_2} e^{-J\tau} b u(\tau) d\tau \\ \cdots\cdots\cdots\cdots \\ \frac{du}{dt}(t_2) \\ \frac{du}{dt}(t_1) \end{bmatrix} \text{ w.r.t. } \lambda. \quad (16)$$

From Equation 15, the unknown λ vector is determined using

$$\lambda = S^{-1} \begin{bmatrix} e^{-J(4\pi)} z(4\pi) \\ \cdots\cdots\cdots\cdots \\ 0 \\ 0 \end{bmatrix}. \quad (17)$$

Solving for λ using the numerical values for this example, and substituting these values into Equation 9 gives the control for the rest-to-rest maneuver for this example as

$$u(t) = 0.0964 - 0.0168t - (0.0093)10^{-t}$$
$$-0.0043e^{(t/4)} \sin\left(\frac{\sqrt{15}}{4}t\right) - 0.0017e^{(t/4)} \cos\left(\frac{\sqrt{15}}{4}t\right) \quad (18)$$

where for this case $a = -1/4$ and $b = \sqrt{15}/4$. Figure 2 is a plot of the control profile (Equation 18) and the position of both masses (m_0 and m_1) for the system using the values given. The rest-to-rest maneuver is completed without any residual vibration.

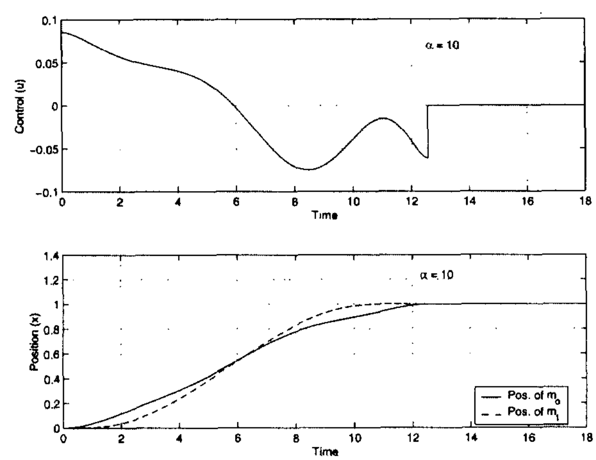

Figure 2: Minimum power/jerk rest-to-rest maneuver.

This section has demonstrated the validity of the weighted minimum power/jerk control determination as applied to a rest-to-rest maneuver of a flexible structure. It can be shown that the weighted minimum power/jerk solution derived here converges to the minimum power and minimum jerk solutions in the limits of the weighting parameter zeta. The next section extends this idea to develop a robust solution when there are errors present in system parameters.

4 Robust Solution - Sensitivity Equations

The goal of this paper is to formulate a control which minimizes a weighted minimum power/jerk cost function while being robust to errors in system parameters. This is done to minimize the residual vibration of the rest-to-rest maneuver when there are errors present in system parameters. To do this, sensitivity equations are derived which represent the sensitivity of the system to model parameters. This procedure can be applied to desensitize the system with respect to the stiffness k, the damping c, or the natural frequency (taking into consideration both errors in mass and stiffness). Here, the sensitivity with respect to the stiffness is of interest (thereby the natural frequency). It will be shown that the control profile obtained with the addition of these sensitivity equations reduces residual vibration when errors in the value of k (stiffness) are present. The equations of motion for the benchmark two-mass/spring/damper problem considered in Section 3 are

$$\ddot{x_0} = \frac{-k}{m_0}x_0 + \frac{k}{m_0}x_1 + \frac{-c}{m_0}\dot{x_0} + \frac{c}{m_0}\dot{x_1} + \frac{u}{m_0} \quad (19)$$

$$\ddot{x_1} = \frac{k}{m_1}x_0 + \frac{-k}{m_1}x_1 + \frac{c}{m_1}\dot{x_0} + \frac{-c}{m_1}\dot{x_1}. \quad (20)$$

By taking the derivative of these two equations with respect to k, the following equations are obtained

$$\frac{d\ddot{x_0}}{dk} + \frac{k}{m_0}\left(\frac{dx_0}{dk} - \frac{dx_1}{dk}\right) + \frac{c}{m_0}\left(\frac{d\dot{x_0}}{dk}\right) + \frac{-c}{m_0}\left(\frac{d\dot{x_1}}{dk}\right) + \frac{x_0}{m_0} - \frac{x_1}{m_0} = 0 \quad (21)$$

and

$$\frac{d\ddot{x_1}}{dk} - \frac{k}{m_1}\left(\frac{dx_0}{dk} - \frac{dx_1}{dk}\right) + \frac{-c}{m_1}\left(\frac{d\dot{x_0}}{dk}\right) + \frac{c}{m_1}\left(\frac{d\dot{x_1}}{dk}\right) - \frac{x_0}{m_1} + \frac{x_1}{m_1} = 0. \quad (22)$$

To simplify formulation while still demonstrating the benefit of this method, the values of the two masses are assumed to be equal as in the previous example ($m_0 = m_1$). This does not have to be the case, but it makes for simple understanding of the procedure. If this assumption is not made, instead of having the single sensitivity equation given in 25, Equations 21 and 22 would represent the sensitivity equations with which the original state space model would be augmented. Using the equal mass assumption, the following equation is obtained using Equations 21 and 22.

$$\frac{dx_0}{dk} = -\frac{dx_1}{dk}. \quad (23)$$

Substituting into Equation 21 gives

$$\frac{d\ddot{x_0}}{dk} + \frac{c}{m}\left(2\frac{d\dot{x_0}}{dk}\right) + \frac{k}{m}\left(2\frac{dx_0}{dk}\right) + \frac{x_0}{m} - \frac{x_1}{m} = 0. \quad (24)$$

Defining a new state variable by $\frac{dx_0}{dk} = x_2$ gives the sensitivity equation, from Equation 24,to be

$$\ddot{x_2} + 2\frac{c}{m}\dot{x_2} + 2\frac{k}{m}x_2 + \frac{x_0}{m} - \frac{x_1}{m} = 0. \quad (25)$$

Once the sensitivity equations are added and the system equations are placed in state space form, the same procedure used to determine the control (from Section 2) can be used. It should be noted that for this particular case, the same equations are obtained if the sensitivity is taken with respect to the damping c. By forcing the sensitivity states to zero at final time, the residual vibration is reduced. The example considered previously in Section 3 will be considered in the next section, this time with the addition of the derived sensitivity equations. A general closed-form solution for the robust weighted minimum power/jerk control can be found by solving Equation 8 with the necessary condition for optimality that the derivative of the control at initial and final time is set equal to zero. Also, the J matrix given in Equation 8 must be augmented with the sensitivity equations to analytically obtain the robust minimum power/jerk closed-form solution given in Equation 26 below. The general closed form solution for the benchmark problem having one rigid body mode and p flexible modes with the addition of the derived sensitivity equations is given as

$$\begin{aligned} u(t) = \lambda_1 + \lambda_2 t + \lambda_3\alpha^t + \lambda_4\alpha^{-t} + \\ \sum_{i=1}^{p}(\lambda_{(1+4i)}e^{-a_i t}\sin(b_i t) + \lambda_{(2+4i)}e^{-a_i t}\cos(b_i t) \\ +\lambda_{(3+4i)}te^{-a_i t}\sin(b_i t) + \lambda_{(4+4i)}te^{-a_i t}\cos(b_i t)) \quad (26) \end{aligned}$$

where a_i is the real part of the i^{th} complex conjugate pole, and b_i is the imaginary part of the i^{th} complex conjugate pole of the system. It should be noted that this general closed-form solution is only valid when the sensitivity is taken with respect to the stiffness (or damping) for the system considered. The next section uses this closed-form solution to obtain the weighted minimum power/jerk control for the example discussed in Section 3, with the inclusion of the sensitivity equations derived here.

5 Numerical Example 2

Using the general closed-form solution given in Equation 26, the robust weighted minimum power/jerk control will be determined for the benchmark problem considered previously with the addition of the sensitivity equation derived in the previous section. Following the same procedure used in Section 3, the control is given as

$$\begin{aligned} u(t) = 0.103461 - 0.019923t - (0.002352)10^{-t} \\ +0.015502e^{(t/4)}\sin\left(\frac{\sqrt{15}}{4}t\right) - 0.000929e^{(t/4)}\cos\left(\frac{\sqrt{15}}{4}t\right) \\ -0.001961e^{(t/4)}t\sin\left(\frac{\sqrt{15}}{4}t\right) - 0.000271e^{(t/4)}t\cos\left(\frac{\sqrt{15}}{4}t\right). \quad (27) \end{aligned}$$

Figure 3 is a plot of the control input for the non-robust and robust solutions when there is a 20 % high error in k ($k = 1.2 * (1/2)$). The figure shows a reduction in the residual vibration due to the error in the system parameter k with the robust solution. Figure 4 is a

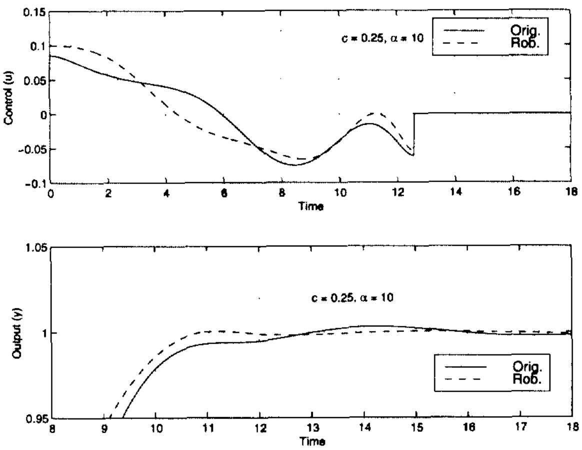

Figure 3: Non-robust and robust control with 20% error in k)

plot of the absolute value of the error in the output (position of m_1) at the specified final time ($t_2 = 4\pi$) vs. the actual system k value using a design k value of 1/2 for both the non-robust and robust solutions. From this figure, the robust solution reduces residual vibration for all actual system k values shown in Figure 4, with the exception that both the non-robust and robust solutions will have zero residual vibration at the design value of $k = 1/2$.

This section has demonstrated the benefit of the derived sensitivity equations when there are errors present in system parameters. The system shown here displays reduced residual vibration with the addition of sensitivity equations. Though this technique may not reduce residual vibration for all actual system k values, it does prove to be a useful method to locally reduce the residual vibration.

6 Conclusions

A systematic procedure to obtain the closed-form solution for the rest-to-rest maneuver of the benchmark problem has been introduced, which minimizes the weighted power/jerk cost function. The concept of sensitivity equations has been introduced which, when added to the system state equations, gives a control which is robust to errors in system parameters. It has been shown that this robust control reduces residual vibration when the actual system parameter is in the vicinity of the design parameter used to derive the control, thus giving a locally robust control. Extensions of this work will include a study of the effect of varying damping as well as the application of this technique to more complicated systems.

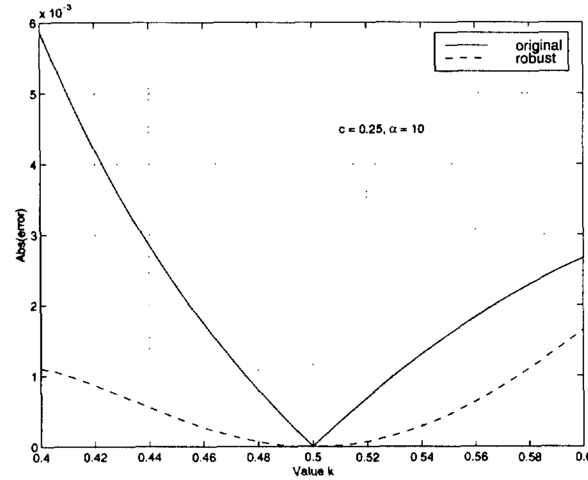

Figure 4: Abs(error) at final time vs. k value for non-robust and robust control

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