Minimax Design of Robust Controllers for Flexible Systems

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Abstract

The focus of this paper is on the design of robust controllers based on the range of expected variation of uncertain parameters from their nominal values. A minimax optimization problem is formulated with the objective of minimizing the maximum value of the cost function over the range of the uncertain parameter. To expedite the optimization process, equations are derived for the gradient of the cost and constraint functions with respect to the parameters of the controller. The proposed technique is illustrated on two examples. The first is a spring-mass-dashpot and the second is a two-mass-spring benchmark problem.

1 Introduction

Control of vibratory structures by filtering the reference input to the system has been addressed by numerous researchers [11], [5], [1], [6] etc. Smith [11] proposed a wave cancellation technique to drive a second order system to its final position in finite time. However, this technique was sensitive to modeling errors. Singer and Seering [5] proposed an approach referred to as *input shaping* which resulted in the same solution as Smith's. They then proposed a simple technique to desensitize the input shaper to modeling errors. This involved design of a sequence of impulses which forced the magnitude of the residual energy and its derivative with respect to damping or natural frequency, to zero. Singh and Vadali [6] arrived at the same results of Singer and Seering [5] by the design of a time-delay filter which cancelled the poles of the system. They also showed that by cascading the time-delay filter designed to cancel the poles of the system, the resulting filter was insensitive to errors in modeled damping and frequency. The idea of locating multiple zeros of a time-delay filter at the estimated location of the poles of the system has been exploited to design robust time-optimal control [7], [3], robust fuel-time optimal control [8], fuel constrained time-optimal control [9] etc. Liu and Singh [2] extended this idea to nonlinear systems undergoing rest-to-rest maneuver, by requiring the sensitivity of the system states with respect to uncertain parameters be zero at the final time.

Techniques to increase the range of uncertain parameters where the residual vibration is below a prespecified amount has been addressed by Singhose et al. [10]. This was referred to as the *extra insensitive input shaper*. Pao et al. [4] proposed including the probability distribution of the uncertain parameters into the design process to arrive at input shapers which weighted the nominal value of the uncertain parameter the most.

The focus of this paper is on the development of a technique to design time-delay filters which minimize the maximum magnitude of the residual vibration over the range in which the uncertain parameter resides. The resulting controller will be referred to as the minimax time-delay controller. Sections 2 and 3 will review the development of the time-delay control and saturating controllers. This will be followed by the development of the minimax time-delay controller in Section 4. The Van Loan identity is used to arrive at equations which represent the gradients of the cost and constraint equations with respect to the parameters of the controller in Section 5. The proposed technique is illustrated on numerical examples in Section 6 and Section 7 summarizes results generated in this paper.

2 Time-Delay Filter

The time-delay control can be considered as a filtering technique which modifies the reference input to a system whose dynamics are characterized by underdamped response (Figure 1). Singh and Vadali [6] propose a single time-delay filter with a transfer function

$$\frac{u(s)}{r(s)} = \frac{A_0}{A_0 + 1} + \frac{e^{-sT}}{A_0 + 1} \tag{1}$$

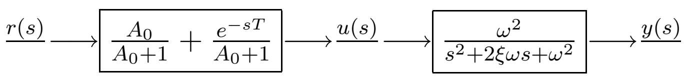

Figure 1: Single Time-Delay Controlled System

to minimize the residual vibration of a single-mode underdamped system and show that to cancel a pair of complex conjugate poles located at $s = -\zeta\omega \pm j\omega\sqrt{1-\zeta^2}$, we require

$$A_0 + e^{\zeta\omega T} cos(\omega\sqrt{1-\zeta^2}T) = 0 \tag{2}$$

$$e^{\zeta\omega T} sin(\omega\sqrt{1-\zeta^2}T) = 0 \tag{3}$$

which results in the solution

$$A_0 = exp(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}) \text{ and } T = \frac{\pi}{\omega\sqrt{1-\zeta^2}}. \tag{4}$$

To address the issue of sensitivity of the pole cancellation time-delay filter, a two time-delay filter is proposed with the constraint that the derivative of the pole cancellation constraint with respect to ζ or ω be forced to zero. The resulting time-delay filter was shown to consist of two single time-delay filters (Equation 1), in cascade. This process of cascading a series of single time-delay filter will progressively increase the insensitivity of the filter to modeling errors. However, the penalty of increased settling time of the response of the system can be significant.

Design of time-delay filters to cancel two or more pairs of stable complex conjugate poles follows the same procedure outlined above. However, the possibility of determining a closed form solution for the parameters of the time-delay filter with a transfer function

$$\frac{u(s)}{r(s)} = A_0 + A_1 e^{-sT_1} + A_2 e^{-sT_2} + ... \tag{5}$$

is remote. To design a multi-mode time-delay filter, we need to solve a set of nonlinear coupled equations derived by substituting

$$s = -\zeta_i\omega_i + j\omega_i\sqrt{1-\zeta_i^2} \text{ where } i = 1, 2, ... \tag{6}$$

into Equation 5 and equating it to zero. The issue of robustness to modeling errors is addressed by cascading time-delay filters designed to cancel the poles of the system, in series.

3 Saturating Controllers

Cost functions such as time, fuel, and weighted fuel-time result in optimal control profiles which are bang-bang or bang-off-bang. These control profiles are very sensitive to uncertainties in modeling and there is thus, a need to design controllers which are insensitive to modeling errors. This has been addressed by Liu and Wei [2], Seering et al. [9], Singh and Vadali [7] etc., where an optimization problem has been formulated which involves the design of a time-delay filter which is required to locate multiple zeros of the time-delay filter at the estimated location of the poles of the system. The constraints for the optimization problem are derived by requiring that the boundary conditions for rest-to-rest or spin-up maneuvers be satisfied for the nominal values of the model parameters. Additional constraints which require the sensitivity of the final states to the uncertain parameters be zero, are included in the optimization problem. For instance, the transfer function of a time-delay filter for the benchmark floating oscillator (Figure 2), is

$$1 - 2e^{-sT_1} + 2e^{-sT_2} - 2e^{-sT_3} + e^{-sT_4}. \tag{7}$$

The time-optimal control profile is generated by driving the time-delay filter with a step input. The constraints for a rest-to-rest maneuver with zero initial conditions can be shown to be

$$-2T_1 + 2T_2 - 2T_3 + T_4 = 0 \tag{8}$$

$$1+2\sum_{i=1}^{3}(-1)^i e^{-\zeta\omega T_i} cos(\omega\sqrt{1-\zeta^2}T_i) + e^{-\zeta\omega T_4} cos(\omega\sqrt{1-\zeta^2}T_4) = 0 \tag{9}$$

$$2\sum_{i=1}^{3}(-1)^i e^{-\zeta\omega T_i} sin(\omega\sqrt{1-\zeta^2}T_i) + e^{-\zeta\omega T_4} sin(\omega\sqrt{1-\zeta^2}T_4) = 0 \tag{10}$$

and

$$T_4^2/2 - (T_4 - T_1)^2 + (T_4 - T_2)^2 - (T_4 - T_3)^2 = k\theta_f \tag{11}$$

where θ_f indicates the total displacement of the rest-to-rest maneuver [7]. The parameters of the time-delay filter are derived by finding a solution which satisfies all the constraints and minimizes T_4. To desensitize the controller to modeling errors, additional time-delays are added to the filter and constraints are derived by forcing the derivatives of Equations 9 and 10 with respect to ω or ζ to be zero.

4 Minimax Time-Delay Control

The time-delay controller and the saturating controllers described above are designed using the nominal values of the model parameters. Robustness is arrived at by studying the sensitivity states evaluated at the nominal value of the system parameters. However, with the knowledge that the uncertain parameters lie within a specified range, it is desirable to design a controller with the worst model in mind. The goal of the optimization problem is to minimize the maximum magni-

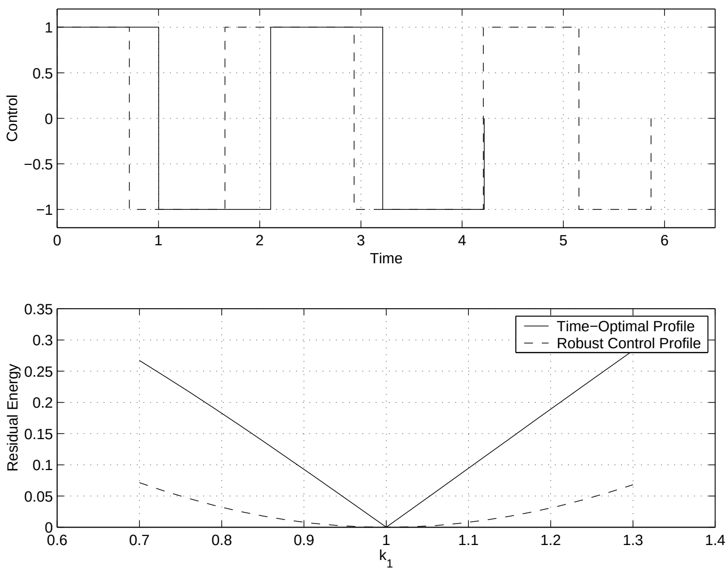

Figure 2: Two-Mass-Spring System

tude of the residual energy over the entire range of the uncertain parameters.

For an asymptotically stable mechanical system undergoing rest-to-rest maneuvers, the model can be represented as

$$M\ddot{y} + C(p)\dot{y} + K(p)y = Dr \tag{12}$$

where M, is a positive definite matrix and K and C are positive semi-definite. K is positive semi-definite when the model of the system includes rigid body modes and is positive definite otherwise. p is a vector of uncertain parameters whose elements satisfy the constraints:

$$p_i^{lb} \le p_i \le p_i^{ub} \tag{13}$$

where p_i^{lb} and p_i^{ub} represent the lower and upper bounds on the parameters respectively. The objective here is to design a time-delay filter which pre-filters the reference input r to the system with the objective of minimizing the maximum value of the residual energy

$$\min_x \max_p F$$
$$F = \frac{1}{2}\dot{y}^T M\dot{y} + \frac{1}{2}(y - y_f)^T K(y - y_f) \tag{14}$$

where x is a vector of parameters which define the robust time-delay filter and y_f corresponds to the final displacement states of the system. The above equation will be referred to as the pseudo-energy function since it is associated with a hypothetical spring whose potential energy is zero when $y = y_f$. The pseudo-energy function is evaluated at the final time, i.e., the end of the maneuver. If K is positive semi-definite, the objective function is

$$\min_x \max_p F$$

$$F = \frac{1}{2}\dot{y}^T M\dot{y}+\frac{1}{2}(y-y_f)^T K(y-y_f)+\frac{1}{2}(y_r-y_{rf})^2 \tag{15}$$

where y_r corresponds to the rigid body displacement and y_{rf} refers to the corresponding desired final displacement. The last term is added to guarantee that the cost function is positive definite. The above formulation weights every point in the uncertain region uniformly. If the designer is provided with information regarding the probability distribution of the uncertain parameters, this information can be included in the optimization problem. For instance, if a Gaussian distribution is assumed for the uncertain parameter, the objective function defined by Equation 14 can be rewritten as

$$\min_x \max_p F$$
$$F = exp(-(p-p_{nom})^T\Gamma^{-1}(p-p_{nom}))\,(\frac{1}{2}\dot{y}^T M\dot{y}+\frac{1}{2}(y-y_f)^T K(y-y_f)) \tag{16}$$

where Γ is the covariance Matrix and p_{nom} is a vector of nominal values of the parameters [4]. Without loss of generality, we can assume that the initial displacement states are zero for the study of rest-to-rest maneuvers. We will derive the necessary equations for the optimization problem based on this assumption.

5 Analytical Gradients

Given a state space model for a system

$$\dot{z} = Az + Bu \text{ where } z = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \in \mathcal{R}^n, u \in \mathcal{R}^1 \tag{17}$$

where u is parametrized as

$$u = A_0 + \sum_{i=1}^{n} A_i \mathcal{H}(t - T_i) \tag{18}$$

where $\mathcal{H}(t - T_i)$ is the Heaviside function, or u is

$$u = 1 + \sum_{i=1}^{n} 2(-1)^i \mathcal{H}(t - T_i) + \mathcal{H}(t - T_{n+1}) \tag{19}$$

for a time-optimal controller, given that $abs(u)$ is less than 1. Assuming a rest-to-rest maneuver, where the initial conditions of the system are zero, the states of the system represented by Equation 17, can be solved for easily, by the technique proposed by Van Loan [12]. To determine the response of a linear system (Equation 17), to a unit step input, construct a matrix

$$P = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tag{20}$$

which is a $\mathcal{R}^{n+1\times n+1}$ matrix. Using the Van Loan identity [12], one can show that

$$M = e^{PT} = \begin{bmatrix} e^{AT} & \int_0^T e^{A(T-\tau)}B\, d\tau \\ 0 & I \end{bmatrix}. \tag{21}$$

It can be seen that the upper right hand term of the matrix M is the convolution integral of the system given by Equation 17 subject to a unit step input. Thus, the value of the states at time T for a unit step input are given by the first n rows of the last column of M. This permits us to calculate the final states for a step input accurately, without numerical simulations. This is very attractive for numerical optimizations, where a significant cost of optimizing dynamical systems is contributed by the numerical simulation of the response of the system. For instance, the response of the system represented by Equation 17 to the input represented by Equation 18 is given by the first n rows of the last column of the matrix

$$\Phi = A_0 e^{PT_n} + \sum_{i=1}^{n} A_i e^{P(T_n - T_i)} \qquad (22)$$

and by the first n rows of the last column of the matrix

$$\Phi = e^{PT_{n+1}} + \sum_{i=1}^{n} 2(-1)^i e^{P(T_{n+1} - T_i)}. \qquad (23)$$

for the time-optimal control (Equation 19).

The optimization algorithms which are used to solve minimax problems are generally gradient based. Thus, the accuracy and the speed of the optimization can be increased by providing analytical gradients to the optimization algorithm. Fortunately, for the time-delay control and the bang-bang and bang-off-bang control profile, closed form equations representing the gradients of the cost and constraints can be easily derived as shown below.

For the optimization algorithm, we require the value of the gradient of the cost F and the constraints, with respect to the controller parameters. For the time-delay controller given by Equation 18, the gradients of F (Equation 14) with respect to A_i and T_i are given by

$$\frac{dF}{dA_i} = (\dot{y}^T M \frac{d\dot{y}}{dA_i} + (y - y_f)^T K \frac{dy}{dA_i}) \qquad (24)$$

and

$$\frac{dF}{dT_i} = (\dot{y}^T M \frac{d\dot{y}}{dT_i} + (y - y_f)^T K \frac{dy}{dT_i}) \qquad (25)$$

To determine $\frac{dy}{dA_i}$, $\frac{d\dot{y}}{dA_i}$ and $\frac{dy}{dT_i}$, $\frac{d\dot{y}}{dT_i}$, we require the derivative of the state Equation 17. The solution of the equation

$$\frac{d\dot{z}}{dA_i} = A\frac{dz}{dA_i} + B\mathcal{H}(t - T_i) \text{ i} = 0,\ 1,\ 2,\ 3,\ ... \qquad (26)$$

can be derived using the Van Loan Identity as described earlier. Similarly the solution of equation

$$\frac{d\dot{z}}{dT_i} = A\frac{dz}{dT_i} - B(A_i \delta(t - T_i)) \text{ i} = 1,\ 2,\ 3,\ ... \qquad (27)$$

where $\delta(.)$ is the dirac delta function, can be shown to be

$$\frac{dz}{dT_i}(T_f) = -A_i exp(A(T_f - T_i))B. \qquad (28)$$

With the analytical gradients, we can expedite the convergence of the optimization algorithm.

6 Numerical Examples

6.1 Spring-Mass-Dashpot

The proposed technique will be illustrated on a rest-to-rest maneuver of a single mode system whose dynamics are defined by the equation

$$m\ddot{y} + c\dot{y} + ky = kr \qquad (29)$$

with the boundary conditions

$$y(0) = \dot{y}(0) = 0,\ \ y(t_f) = 1, \dot{y}(t_f) = 0 \qquad (30)$$

where t_f is the maneuver time.

First, a minimax time-delay controller will be designed assuming that only k is uncertain and satisfies the constraint $0.7 \leq k \leq 1.3$, where the nominal value of $k = 1$, $m = 1$, and $c = 0.2$. The form of the transfer function for the minimax time-delay controller is chosen to be

$$A_0 + A_1 e^{-sT_1} + A_2 e^{-sT_2} \qquad (31)$$

which is identical to the robust time-delay controller [6]. The optimization problem can be stated as the determination of A_0, A_1, A_2, T_1 and T_2 of the time-delay filter so as to

$$\min_{T_i, A_i} \max_{k} \left(\frac{1}{2}m\dot{y}^2 + \frac{1}{2}k(y-1)^2\right) \qquad (32)$$

evaluated at T_2. The initial guess for the minimax optimization problem is the robust time-delay filter. To determine the parameters of the robust time-delay filter, we need to solve for the non-robust time-delay filter first, which is

$$0.5783 + 0.4217e^{-3.1574s}. \qquad (33)$$

With the knowledge that two non-robust filters in cascade will force the derivative of the square root of the pseudo-energy to be zero at the nominal value of the system parameters resulting in smaller magnitude of residual vibration in the vicinity of the nominal parameters as illustrated in Figure 3 (dash-dot line), the transfer function of the robust time-delay controller can be shown to be

$$0.3344 + 0.4877e^{-3.1574s} + 0.1788e^{-6.3148s}. \qquad (34)$$

The parameters of the time-delay filter (Equation 34), will be used as initial guesses for the minimax algorithm. The optimization toolbox of MATLAB is used

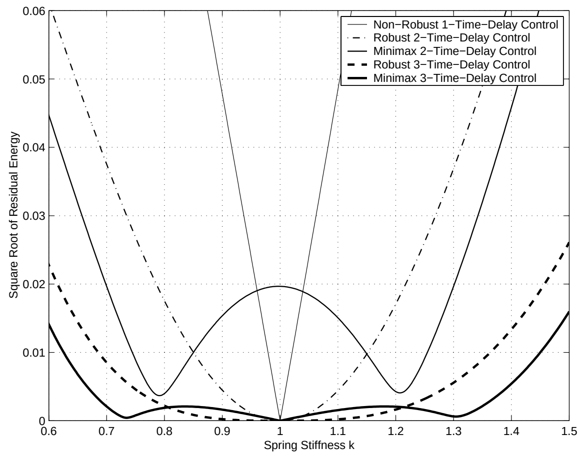

Figure 3: Residual Vibration Distribution

to solve the minimax optimization problem which results in the transfer function

$$0.3452 + 0.4730e^{-3.1703s} + 0.1818e^{-6.2060s}. \quad (35)$$

Figure 3 (solid line) illustrates the variation of the residual energy of the system as a function of the uncertain parameter k. It can be seen that the maximum magnitude of the residual energy in the range of the uncertain parameters occurs at the bounding limits (k = 0.7, k = 1.3) and near the nominal value of k. It is also clear that the maximum magnitude of the residual energy is significantly smaller than that resulting from the robust time-delay filter defined by Equation 34 over the entire range of k. However, at the nominal value of k, the minimax solution has the largest magnitude of residual vibration.

Notwithstanding that the maximum magnitude of the residual vibration over the range of possible value of k has been minimized, the fact that the residual vibration at the nominal value of k is the maximum, is a drawback of this controller. To address the aforementioned disadvantage, an additional constraint is included into the minimax optimization problem which requires the magnitude of the residual vibration to be zero at the nominal value of the uncertain parameter. The added constraint necessitates addition of a time-delay to the time-delay filter defined by Equation 35. The transfer function of the minimax time-delay controller with the constraint to force the residual vibration to be zero at the nominal value of k can be shown to be

$$0.205+0.414e^{-3.17s}+0.302e^{-6.33s}+0.079e^{-9.49s}. \quad (36)$$

Figure 3 illustrate the distribution of the residual energy of the time-delay filter designed by cascading three non-robust time-delay filters (thick dash line) and the minimax time-delay filter (thick solid line).

The second example considered for the illustration of the proposed technique is the benchmark two mass-spring system. Unlike the first example, this system is characterized by rigid body modes and the sum of the kinetic and potential energy is not a positive definite function and therefore the energy of the system is augmented with a term which reflects the energy stored in a virtual spring whose potential energy is zero when the masses are at the final desired positions. The equations of motion of the floating oscillator are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{y_1} \\ \ddot{y_2} \end{bmatrix} u + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad (37)$$

where u is bounded by the constraint

$$-1 \le u \le 1. \quad (38)$$

The objective of the optimization problem is to design a control profile for a rest-to-rest maneuver which satisfies the boundary conditions

$$y_1 = y_2 = \dot{y_1} = \dot{y_2} = 0|_{t=0}, \text{ and } y_1 = y_2 = 1, \dot{y_1} = \dot{y_2} = 0|_{t=t_f}. \quad (39)$$

The optimal control profile is parameterized as Equation 19. A minimax problem is formulated to solve for the maneuver and switch times T_i, assuming that k is uncertain and satisfies the constraint $0.7 \le k \le 1.3$ where the cost function is

$$\min_{T_i} \max_{k} (\frac{1}{2} \begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix}^T \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} +$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{1}{2}(y_1 - 1)^2) \quad (40)$$

Assuming $n = 3$, in Equation 19 and solving the minimax problem with the constraint that the magnitude of the pseudo-energy function (Equation 40) be zero at $k = 1$, the nominal value of the uncertain parameter, we arrive at the time-optimal control profile

$$u = 1 - 2\mathcal{H}(t - 1.0027) + 2\mathcal{H}(t - 2.1089)$$
$$-2\mathcal{H}(t - 3.2151) + \mathcal{H}(t - 4.2178). \quad (41)$$

Solving the minimax control problem without enforcing the requirement that the residual energy should be zero at the nominal value of the uncertain parameter results in the control profile

$$u = 1 - 2\mathcal{H}(t - 0.9430) + 2\mathcal{H}(t - 2.0571)$$
$$-2\mathcal{H}(t - 3.1713) + \mathcal{H}(t - 4.1143). \quad (42)$$

Figure 4 illustrates the variation of the residual energy of the floating oscillator as a function of k for the 3 and 5 switch control profiles. It is clear that the maximum magnitude of the residual energy of the control profile given by Equation 42 (dash line) is smaller than that given by Equation 41 (solid line) in the range of

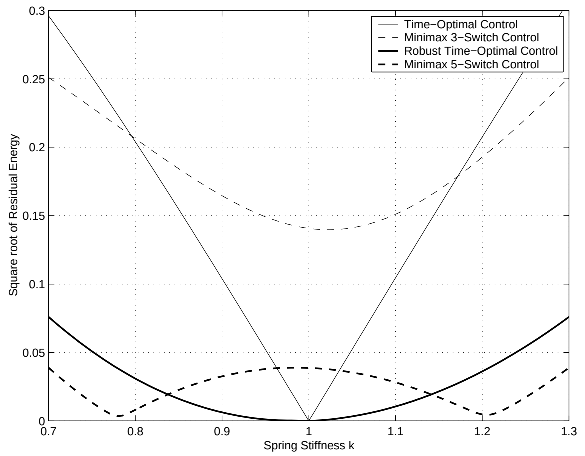

Figure 4: Residual Energy Distribution

uncertain k. It is to be noted that the maximum magnitude of the residual vibration occurs at the end of the interval of uncertainty, i.e., $k = 0.7$ and $k = 1.3$. However, the residual energy is not zero at $k = 1$. A 5-switch control profile is selected next to reduce the maximum magnitude of residual energy. The minimax optimization problem is solved again with and without the constraint that the residual energy for $k = 1$ should be zero. The resulting control profile is

$$u = 1-2\mathcal{H}(t-0.7181)+2\mathcal{H}(t-1.6715)-2\mathcal{H}(t-2.9526)+ 2\mathcal{H}(t-4.2370)-2\mathcal{H}(t-5.1851)+\mathcal{H}(t-5.8944). \quad (43)$$

The thick solid line of Figure 4 illustrates the distribution of the residual energy with constraint that the residual energy be zero at $k = 1$. A point to note is that the optimization algorithm resulted in a control profile which forces the slope of the energy distribution curve to be zero at $k = 1$, without the explicit requirement of that constraint as in the work of Singh and Vadali [7]. The thick dash line of Figure 4 illustrates that the elimination of the constraint that the residual energy be zero at $k = 1$, results in a significant reduction of the maximum magnitude of residual energy in the uncertain range which again occur at the ends of the uncertain region. One can note again that the residual energy is not zero over the entire interval of the uncertain parameter k and the control profile is given by the equation

$$u = 1-2\mathcal{H}(t-0.7256)+2\mathcal{H}(t-1.6909)-2\mathcal{H}(t-2.9595)+ 2\mathcal{H}(t-4.2281)-2\mathcal{H}(t-5.1934)+\mathcal{H}(t-5.9190). \quad (44)$$

7 Conclusions

This paper presents a technique for the design of robust controllers which minimize the maximum magnitude of the cost function over the uncertain interval. Closed form equations for the gradients of the cost and constraint functions with respect to the parameters to be optimized for, are derived, which aid in the numerical optimization. The proposed technique is illustrated on two examples. The first is a spring-mass-dashpot system and involves the design of a time-delay filter to minimize the maximum magnitude of residual vibration for a unit step input. The second example, is the design of a robust bang-bang controller for rest-to-rest maneuvers of the two-mass spring benchmark problem.

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