Controller Design for Flexible Systems with Friction: Pulse Amplitude Control

Jae-Jun Kim * Tarunraj Singh †
State University of New York at Buffalo, Buffalo, NY 14260

ABSTRACT

In this paper, a technique to determine the pulse amplitude modulated control input for a frictional system is proposed. A user specified pulse width is used to initiate the motion so as to permit the system to coast to the desired final position after the final pulse, with zero residual vibrations. The proposed technique is illustrated on the floating oscillator where the first mass is under the influence of friction. Numerical simulation illustrates the effectiveness of the proposed technique.

INTRODUCTION

Friction is highly nonlinear in the low velocity region and when there is a velocity reversal. For precise positioning and pointing systems, difficulty in control arises near the desired final position because of stiction. Conventional PD and PID controllers are known to cause steady state error and hunting.[1] Yang and Tomizuka[2] developed the adaptive pulse width control technique for rigid body systems. The pulse width control can avoid the problems of hunting and velocity reversals by allowing the system to coast towards the desired position. Successive pulses are applied until the desired position is reached while unknown parameters are adapted at the end of the each pulse. With the static and Coulomb friction model used in,[2] the friction force is considered constant because of the guaranteed unidirectional motion of the system. Rathbun[3] extended the pulse width control to the flexible two mass spring damper system. He used the single pulse to study the stability bound on the pulse widths. Although the controller is stable, the flexible states excited by the input pulse will result in undesirable residual vibration. If the damping is small, the settling time will increase which will increase the total maneuver time. In this paper, pulse amplitude modulated control profile is proposed to eliminate the unwanted vibration at the end of the maneuver. If positive velocity of the frictional body is maintained during the maneuver, the friction force can be considered as a biased input and linear design techniques can be used. The various control profile has been found via Linear programming[5] with the positive velocity of the frictional body. In the proposed technique, the time-delay filtering technique[4] is utilized for vibration suppression. If stiction occurs during the maneuver, the control profile has to be modified as illustrated in Section . Numerical simulations are performed to verify the proposed controllers.

PROBLEM FORMULATION

The floating oscillator under the influence of friction is illustrated in Figure 1, where m_1, m_2 are the first and second mass, k is the spring constant, u the control input, f the friction force and x_1, x_2 are the positions of the first and second mass. The equation

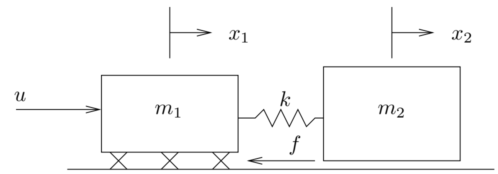


Fig. 1 Floating Oscillator under Friction

of motion of the system can be written as

$$M\underline{\ddot{x}} + K\underline{x} = D(u - f) \tag{1}$$

where M, K, and D are

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2}$$

The friction force is modelled as a static nonlinear function of the velocity which accounts for static and coulomb friction. The friction model can be represented as:

$$f(\underline{x}, u) = \begin{cases} f_c sgn(\dot{x}_1) & \text{if } \dot{x}_1 \neq 0 \\ f_s sgn(u_s) & \text{if } \dot{x}_1 = 0 \text{ and } u_s > f_s \\ u_s & \text{if } \dot{x}_1 = 0 \text{ and } u_s \leq f_s \end{cases} \tag{3}$$

where, f_s is the static friction, f_c is the coulomb friction and u_s is the sum of the forces applied to the first mass, which is

$$u_s = u + k(x_2 - x_1) \tag{4}$$

If the first mass velocity never goes to zero and stays positive during the maneuver, the friction force for a rest-to-rest maneuver becomes

$$f = f_c\left[1 - H(t - T_f)\right] \tag{5}$$

where, $H(\cdot)$ is the Heaviside step function and T_f is the final time. With this friction model, Equation 1 becomes

$$M\underline{\ddot{x}} + K\underline{x} = D\left\{u - f_c\left[1 - H(t - T_f)\right]\right\} \tag{6}$$

It is more convenient to study the floating oscillator system if the decoupled equation of motion is used. Define new decoupled states $\underline{z} = [\theta, q]^T$, where θ and q denotes the rigid and flexible body states of the system. The transformation matrix V can be found from the eigenvectors of the system, which decouples the system with the relationship $\underline{x} = V\underline{z}$. The decoupled state equation becomes

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \theta \\ q \end{bmatrix} = \begin{bmatrix} \frac{1}{m_1+m_2} \\ -\frac{1}{m_1+m_2} \end{bmatrix} \left\{u - f_c\left[1 - H(t - T_f)\right]\right\} \tag{7}$$

*Graduate Student: jjk@eng.buffalo.edu
†Associate Professor: tsingh@eng.buffalo.edu

where, matrix V and natural frequency ω are

$$V = \begin{bmatrix} 1 & -\frac{m_2}{m_1} \\ 1 & 1 \end{bmatrix}, \quad \omega = \sqrt{\frac{k(m_1+m_2)}{m_1 m_2}} \tag{8}$$

For the rest-to-rest maneuver problem, the boundary conditions are

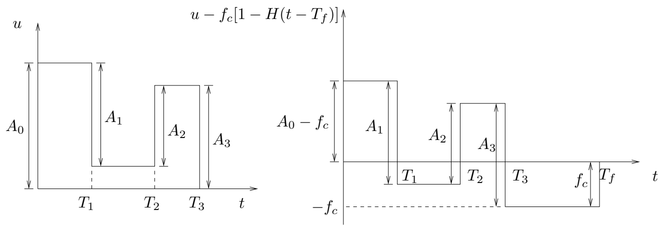
$$\begin{array}{ll} x_1(0) = x_2(0) = 0 & x_1(T_f) = x_2(T_f) = d \\ \dot{x_1}(0) = \dot{x_2}(0) = 0 & \dot{x_1}(T_f) = \dot{x_2}(T_f) = 0 \end{array} \tag{9}$$

where, d is the desired position at T_f. The equivalent boundary conditions in decoupled states become

$$\begin{array}{ll} \theta(0) = \dot{\theta}(0) = 0 & \theta(T_f) = d,\, \dot{\theta}(T_f) = 0 \\ q(0) = \dot{q}(0) = 0 & q(T_f) = \dot{q}(T_f) = 0 \end{array} \tag{10}$$

POLE-ZERO CANCELLATION

In our development, a three pulse profile is initially assumed as shown in Figure 2(a). The pulse widths are selected by the user



(a) Three Pulse Input (b) Coulomb Friction Biased Input

Fig. 2 Input Profile

and the pulse amplitudes are determined to satisfy the boundary conditions. Since positive velocity of the first mass is assumed, the new input to the linear system is biased by the magnitude of Coulomb friction as shown in Figure 2(b). The Coulomb friction biased input can be written as

$$\begin{array}{r} u(t) - f_c[1 - H(t-T_f)] = (A_0 - f_c) - A_1 H(t - T_1) \\ +A_2 H(t-T_2) - A_3 H(t-T_3) + f_c H(t-T_f) \end{array} \tag{11}$$

Because the control input in Figure 2(a) should be zero for $t \geq T_3$, the following is true.

$$A_0 - A_1 + A_2 - A_3 = 0 \tag{12}$$

In order to eliminate the vibration at the end of the maneuver, zeros of the input should cancel the flexible mode poles of the system.[4] To cancel the flexible mode poles, Equation 11 should satisfy the following equation.

$$\begin{array}{r} (A_0 - f_c) - A_1 e^{-sT_1} + A_2 e^{-sT_2} \\ -A_3 e^{-sT_3} + f_c e^{-sT_f} \,|_{s=\pm j\omega} = 0 \end{array} \tag{13}$$

Equation 13 can be rewritten as:

$$A_0 - A_1 \cos\omega T_1 + A_2 \cos\omega T_2 - A_3 \cos\omega T_3 = f_c(1 - \cos\omega T_f) \tag{14}$$

$$-A_1 \sin\omega T_1 + A_2 \sin\omega T_2 - A_3 \sin\omega T_3 = -f_c sin\omega T_f \tag{15}$$

The displacement of the rigid body at the final time is sum of the rigid body displacement at $t = T_3$ and the coasting displacement such that

$$\theta(T_f) = \theta(T_3) + \frac{m_1+m_2}{2f_c}\left[\dot{\theta}(T_3)\right]^2 = d \tag{16}$$

where, $\theta(T_3)$ and $\dot{\theta}(T_3)$ are found by solving the rigid body differential equation as follows.

$$\begin{array}{l} \theta(T_3) = \frac{1}{2(m_1+m_2)}\left[A_0(T_3)^2 - A_1(T_3-T_1)^2 + A_2(T_3-T_2)^2 \right. \\ \qquad \left. -f_c(T_3)^2\ \right] \\ \dot{\theta}(T_3) = \frac{1}{m_1+m_2}\left[A_0 T_3 - A_1(T_3-T_1) + A_2(T_3-T_2) - f_c T_3\right] \end{array} \tag{17}$$

The final time can be found by adding the coasting time to T_3. Since, satisfying Equations 14 and 15 is equivalent to the flexible states being zero at the final time, the coasting time is found by solving the rigid body equation for $t > T_3$ and equating the velocity of the rigid body to be zero. The final time becomes

$$T_f = T_3 + \frac{\dot{\theta}(T_3)(m_1+m_2)}{f_c} = \frac{1}{f_c}(A_1T_1 - A_2T_2 + A_3T_3) \tag{18}$$

An approach for solving for the final time will be presented in the next section.

ZERO-RESIDUAL VIBRATION

If the flexible motion states $q(T_f)$ and $\dot{q}(T_f)$ are forced to zero at the final time, residual vibration will be eliminated. Since the final time in Equation 18 is a function of pulse amplitudes, Equation 14 and 15 are difficult to solve. To solve this problem, the states of the flexible mode at $t = T_3$ that will force the flexible motion to be zero at the final time are derived. Solutions of the flexible mode equation at $t = T_3$ are

$$\begin{array}{l} q(T_3) = -\frac{1}{\omega^2(m_1+m_2)}\left[-f_c - A_0\cos\omega T_3 + A_1\cos\omega(T_3-T_1) \right. \\ \qquad \left. -A_2\cos\omega(T_3-T_2) + A_3 + f_c\cos\omega T_3\right] \\ \dot{q}(T_3) = -\frac{1}{\omega(m_1+m_2)}\left[A_0\sin\omega T_3 - A_1\sin\omega(T_3-T_1) \right. \\ \qquad \left. +A_2\sin\omega(T_3-T_2) - f_c\sin\omega T_3\right] \end{array} \tag{19}$$

The equation of motion of the flexible mode for the coasting period with the initial condition of $q(T_3)$ and $\dot{q}(T_3)$ becomes

$$\ddot{q}_c + \omega^2 q_c = \frac{f_c[1 - H(t-T_c)]}{m_1+m_2} \tag{20}$$

where, q_c is the flexible mode state for the coasting period and the coasting time $T_c = T_f - T_3$. The solution to Equation 20 becomes

$$\begin{array}{l} q_c(t) = \frac{f_c}{m_1+m_2}\left[\left(\frac{1}{\omega^2} - \frac{\cos\omega t}{\omega^2}\right) - \left(\frac{1}{\omega^2} - \frac{\cos\omega(t-T_c)}{\omega^2}\right)H(t-T_c)\right] \\ \qquad +q(T_3)\cos\omega t + \frac{\dot{q}(T_3)\sin\omega t}{\omega} \\ \dot{q}_c(t) = \frac{f_c}{m_1+m_2}\left(\frac{\sin\omega t}{\omega}\right) - q(3\Delta t)\omega\sin\omega t + \dot{q}(3\Delta t)\cos\omega t \end{array} \tag{21}$$

At $t = T_c$, the flexible motion should be eliminated. By substituting T_c into Equation 21 and equating them to zero, the flexible states at $t = T_3$ which will force the flexible states to zero at the final time are

$$\begin{bmatrix} q(T_3) \\ \dot{q}(T_3) \end{bmatrix} = \begin{bmatrix} -\dfrac{f_c(\cos\omega T_c - 1)}{\omega^2(m_1+m_2)} \\ -\dfrac{f_c \sin\omega T_c}{\omega(m_1+m_2)} \end{bmatrix} \tag{22}$$

The flexible states at $t = T_3$ shown in Equation 22 will force the flexible motion to be eliminated at the end of the maneuver if T_c is known. However, the total maneuver time is a function of pulse amplitudes and therefore, an iterative approach is used to find the total maneuver time and T_c. Rewriting the constraint Equations 12 and 19 in terms of A_1, A_2 and A_3, the constraint equations in matrix form become

$$\begin{bmatrix} -1 & 1 & -1 \\ -\cos\omega(T_3-T_1) & \cos\omega(T_3-T_2) & -1 \\ -\sin\omega(T_3-T_1) & \sin\omega(T_3-T_2) & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -\cos\omega T_3 \\ -\sin\omega T_3 \end{bmatrix} A_0 + \begin{bmatrix} 0 \\ f_c\cos\omega T_3 - f_c + \omega^2(m_1+m_2)q(T_3) \\ f_c\sin\omega T_3 - \omega(m_1+m_2)\dot{q}(T_3) \end{bmatrix} \tag{23}$$

To find initial values for the input pulse amplitudes and total maneuver time, solve Equation 23 for A_1, A_2, and A_3 in terms of A_0 by letting $q(T_3) = \dot{q}(T_3) = 0$. By substituting A_1, A_2, and A_3 into the rigid body constraint in Equation 16, A_0 can be found. Once the pulse amplitudes are found, the initial total maneuver time is found by substituting the pulse amplitudes into Equation 18. With this initial T_f, flexible states at $t = T_3$ are computed using Equation 22. The new pulse amplitudes and total maneuver time is calculated with this new flexible states at $t = T_3$. This procedure is repeated until the flexible states and total maneuver time converge.

NUMERICAL SIMULATION

Numerical simulations are used to illustrate the performance of the proposed controllers. The parameter values used in the simulation are shown in Table 1. The first simulation is performed

Table 1 Parameters Used in the Simulation

symbol	description	value
m_1	mass 1	80 Kg
m_2	mass 2	100 Kg
ω	natural frequency	50 rad/sec
u_p	peak input	1000 N
f_s	static friction	137 N
f_c	Coulomb friction	111 N

with the initial and final states of

$$\begin{array}{ll} x_1(0) = x_2(0) = 0 & x_1(T_f) = x_2(T_f) = 0.1 \\ \dot{x}_1(0) = \dot{x}_2(0) = 0 & \dot{x}_1(T_f) = \dot{x}_2(T_f) = 0 \end{array} \tag{24}$$

The pulse widths are chosen such that $T_1 = 0.1$ sec, $T_2 = 0.2$ sec and $T_3 = 0.3$ sec. The resulting control input and responses are shown in Figure 3. The solid line represents the first mass and the dashed line represents the second mass. It is shown that the

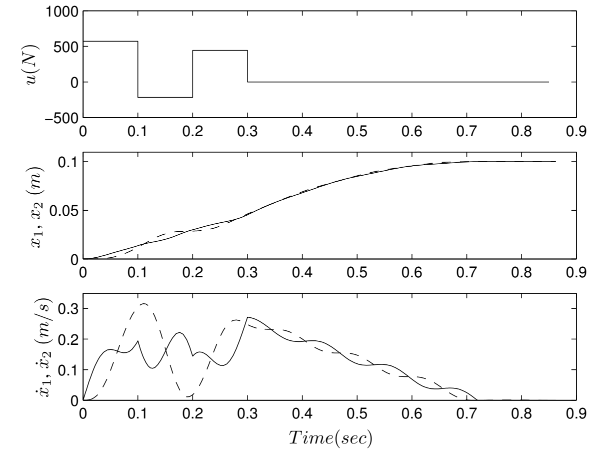


Fig. 3 Three Pulse Control Input and Responses ($d = 0.1$)

first mass velocity is always positive and therefore unidirectional friction force is applied to the system during the maneuver. The system begins to coast at $t = 0.3$ sec and the undesirable vibration is eliminated at the final time. The flexible state at $t = 0.3$ are forced such that at the final time the flexible states become zero, which result in zero residual vibration.

Figure 4 shows the pulse input amplitude change due to the changes in the command displacement. The pulse widths are selected the same as previous example such that $T_1 = 0.04$ sec, $T_2 = 0.08$ sec and $T_3 = 0.12$ sec. The velocity of the mass becomes zero during the maneuver if $d < 0.0039$ m which corresponds to the lower displacement bound with the chosen pulse widths. Because of the control input saturation, different pulse widths are selected for $d > 0.278$ m. The plot of the pulse input amplitudes for different natural frequency values are shown in Figure 5 with the same input pulse widths. It is also shown in Figure 5 that there are no feasible solutions in the frequency range from 76 to 79 rad/sec. This is because the matrix in Equation 23 becomes singular when the switching time is chosen such that:

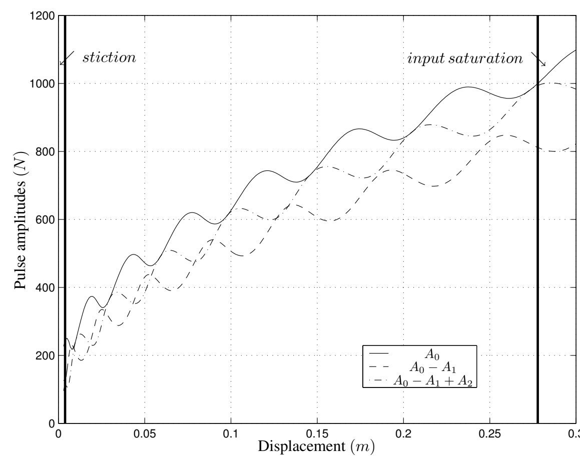


Fig. 4 Pulse Amplitudes vs. Displacement

$$\omega(T_3 - T_2) = 2n\pi \quad \text{or} \quad \omega(T_3 - T_1) = 2n\pi \tag{25}$$

where, n is a positive integer. Therefore different pulse widths should be selected for designing a controller if the pulse widths chosen make the condition number of the matrix in Equation 23 very large.

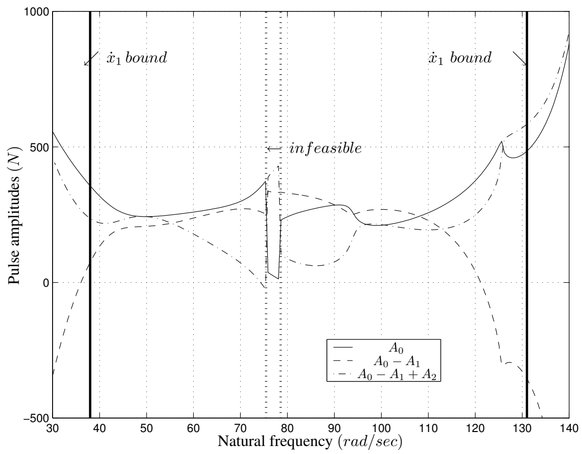


Fig. 5 Pulse Amplitudes vs. Natural Frequency

SYSTEMS WITH STICTION

When the command displacement is small, it may not be feasible to find a solution which will guarantee positive velocity of the first mass. If the velocity becomes zero during the maneuver, stiction might occur. If stiction is considered in the controller design, the control profile is assumed to be as shown in Figure 6. In the

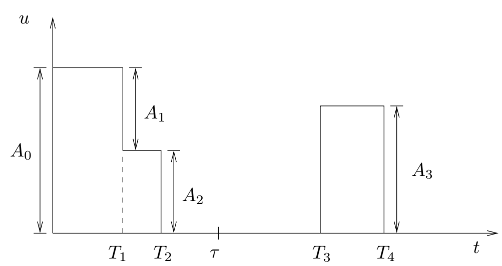


Fig. 6 Input Profile with stiction

figure, τ is the time when the velocity of the first mass becomes zero and the controller assumes that the first mass stays stuck for $\tau \leq t \leq T_3$. During stiction, the sum of the applied forces to the first mass should be less than or equal to the static friction value. Since the input is zero during the stiction, the condition of staying stuck becomes

$$k[x_2(t) - x_1(t)] \leq f_s \quad (\tau \leq t \leq T_3) \tag{26}$$

If the above condition is not met, the spring force has to be compensated to force the first mass to stay stuck for $\tau \leq t \leq T_3$. Spring force compensation is considered later in the section. Define the relative displacement and velocity at $t = T_3$ as

$$\eta_0 = x_2(T_3) - x_1(T_3), \quad \dot{\eta}_0 = \dot{x}_2(T_3) \tag{27}$$

First, η_0, $\dot{\eta}_0$ and A_3 are determined which will satisfy the boundary condition without the residual vibration at the final time. The flexible states at $t = T_4$ in terms of η_0 and $\dot{\eta}_0$ are

$$\begin{aligned} q(T_4) &= -\tfrac{1}{\omega^2(m_1+m_2)}[A_3 - f_c - A_3\cos\omega(T_4-T_3) \\ &\quad + f_c\cos\omega(T_4-T_3)] + \tfrac{m_1\cos\omega(T_4-T_3)}{m_1+m_2}\eta_0 + \tfrac{m_1\sin\omega(T_4-T_3)}{\omega(m_1+m_2)}\dot{\eta}_0 \\ \dot{q}(T_4) &= -\tfrac{1}{\omega(m_1+m_2)}[A_3\sin\omega(T_4-T_3) - f_c\sin\omega(T_4-T_3)] \\ &\quad - \tfrac{m_1\omega\sin\omega(T_4-T_3)}{m_1+m_2}\eta_0 + \tfrac{m_1\cos\omega(T_4-T_3)}{m_1+m_2}\dot{\eta}_0 \end{aligned} \tag{28}$$

Equation 28 is solved for η_0 and $\dot{\eta}_0$.

$$\begin{aligned} \begin{bmatrix} \eta_0 \\ \dot{\eta}_0 \end{bmatrix} &= \begin{bmatrix} \frac{\cos\omega(T_4-T_3)-1}{m_1\omega^2} \\ \frac{\sin\omega(T_4-T_3)}{m_1\omega} \end{bmatrix} A_3 + \begin{bmatrix} \frac{f_c[1-\cos\omega(T_4-T_3)]}{m_1\omega^2} \\ -\frac{f_c\sin\omega(T_4-T_3)}{m_1\omega} \end{bmatrix} \\ &\quad + \frac{m_1+m_2}{m_1}\begin{bmatrix} \cos\omega(T_4-T_3) & -\frac{\sin\omega(T_4-T_3)}{\omega} \\ \omega\sin\omega(T_4-T_3) & \cos\omega(T_4-T_3) \end{bmatrix}\begin{bmatrix} q(T_4) \\ \dot{q}(T_4) \end{bmatrix} \end{aligned} \tag{29}$$

The displacement boundary condition is

$$\theta(T_f) = \theta(T_4) + \frac{m_1+m_2}{2f_c}[\dot{\theta}(T_4)]^2 = d \tag{30}$$

where, $\theta(T_4)$ and $\theta(\dot{T}_4)$ are

$$\begin{aligned} \theta(T_4) &= x_{1s} + \tfrac{A_3-f_c}{2(m_1+m_2)}(T_4-T_3)^2 + \tfrac{m_2}{m_1+m_2}[\eta_0 + \dot{\eta}_0(T_4-T_3)] \\ \dot{\theta}(T_4) &= \tfrac{A_3-f_c}{m_1+m_2}(T_4-T_3) + \tfrac{m_2}{m_1+m_2}\dot{\eta}_0 \end{aligned} \tag{31}$$

x_{1s} is the first mass position under stiction, which is unknown. Therefore, an iterative procedure is required starting with the initial x_{1s}. For small displacements, an initial x_{1s} close to half of the displacement to be moved makes a good starting point. Then, η_0, $\dot{\eta}_0$ and A_3 can be computed similar to the approach for the non stiction case. Assuming $q(T_4) = \dot{q}(T_4) = 0$ initially, solve η_0 and $\dot{\eta}_0$ in terms of A_3 from Equation 29. A_3 is found from the displacement boundary condition in Equation 30. With the initial solutions of η_0, $\dot{\eta}_0$ and A_3, the flexible body states at $t = T_4$ which will eliminate the residual vibration are found as follows.

$$q(T_4) = -\frac{f_c(\cos\omega T_c - 1)}{\omega^2(m_1+m_2)} \tag{32}$$

$$\dot{q}(T_4) = -\frac{f_c\sin\omega T_c}{\omega(m_1+m_2)} \tag{33}$$

where, the coasting time T_c is

$$T_c = \frac{1}{f_c}[(A_3 - f_c)(T_4 - T_3) + m_2\dot{\eta}_0] \tag{34}$$

This procedure is repeated until the flexible states at $t = T_4$ and total maneuver time converge. Once η_0 and $\dot{\eta}_0$ are determined, the state constraints at $t = \tau$ should be found to solve for A_0, A_1 and A_2. However, τ is a function of the input pulse amplitudes which are not yet determined. Therefore an iterative approach is applied again with the initial assumption of τ. Because the stiction occurs between τ and T_3, the system behaves like a single mass harmonic oscillator such that

$$m_2\ddot{\psi} + k\psi = 0 \tag{35}$$

with the initial and final conditions

$$\begin{array}{ll} \psi_0 = x_2(\tau) - x_1(\tau) & \psi_f = x_2(T_3) - x_1(T_3) = \eta_0 \\ \dot{\psi}_0 = \dot{x}_2(\tau) & \dot{\psi}_f = \dot{x}_2(T_3) = \dot{\eta}_0 \end{array} \tag{36}$$

Since final states are specified, solving Equation 35 for ψ_0 and $\dot{\psi}_0$ yields

$$\begin{bmatrix} \psi_0 \\ \dot{\psi}_0 \end{bmatrix} = \begin{bmatrix} \cos\omega_s(T_3-\tau) & -\dfrac{\sin\omega_s(T_3-\tau)}{\omega_s} \\ \omega_s \sin\omega_s(T_3-\tau) & \cos\omega_s(T_3-\tau) \end{bmatrix} \begin{bmatrix} \eta_0 \\ \dot{\eta}_0 \end{bmatrix} \tag{37}$$

where $\omega_s = \sqrt{k/m_2}$. Then, the flexible states at $t = \tau$ are found in terms of ψ_0 and $\dot{\psi}_0$.

$$\begin{aligned} q(\tau) &= \frac{m_1}{m_1+m_2}\psi_0 \\ \dot{q}(\tau) &= \frac{m_1}{m_1+m_2}\dot{\psi}_0 \end{aligned} \tag{38}$$

Constraint equations required to solve for A_0, A_1 and A_2 are

$$A_0 - A_1 - A_2 = 0 \tag{39}$$

$$\begin{aligned} q(\tau) = \tfrac{1}{\omega^2(m_1+m_2)} & [A_0\cos\omega\tau - A_1\cos\omega(\tau-T_1) \\ & -A_2\cos\omega(\tau-T_2) + f_c(1-\cos\omega\tau)\ ] \end{aligned} \tag{40}$$

$$\begin{aligned} \dot{q}(\tau) = -\tfrac{1}{\omega(m_1+m_2)} & [A_0\sin\omega\tau - A_1\sin\omega(\tau-T_1) \\ & -A_2\sin\omega(\tau-T_2) - f_c\sin\omega\tau\ ] \end{aligned} \tag{41}$$

Because $q(\tau)$ and $\dot{q}(\tau)$ should satisfy Equation 38, we have the following simultaneous equation.

$$\begin{aligned} \begin{bmatrix} 1 & -1 & -1 \\ \cos\omega\tau & -\cos\omega(\tau-T_1) & -\cos\omega(\tau-T_2) \\ \sin\omega\tau & -\sin\omega(\tau-T_1) & -\sin\omega(\tau-T_2) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} \\ = \begin{bmatrix} 0 \\ m_1\omega^2\psi_0 - f_c(1-\cos\omega\tau) \\ -m_1\omega\dot{\psi}_0 + f_c\sin\omega\tau \end{bmatrix} \end{aligned} \tag{42}$$

The resulting A_0, A_1 and A_2 should satisfy the condition that the velocity of the first mass at $t = \tau$ becomes zero. The velocity of the first mass at $t = \tau$ can be written as:

$$\dot{x}_1(\tau) = \dot{\theta}(\tau) - \frac{m_2}{m_1}\dot{q}(\tau) = 0 \tag{43}$$

where, $\dot{q}(\tau)$ is found from Equation 38 and $\theta(\tau)$ is found from the following equation.

$$\theta(\tau) = \frac{1}{2(m_1+m_2)}[A_0\tau^2 - A_1(\tau-T_1)^2 - A_2(\tau-T_2)^2 - f_c\tau^2] \tag{44}$$

If the velocity constraint of the first mass at $t = \tau$ is violated, the τ should be updated. Using the Newton-Raphson method, the new τ is updated by the following relationship.

$$\tau_{new} = \tau_{old} - \frac{\dot{x}_1(\tau_{old})}{\ddot{x}_1(\tau_{old})} \tag{45}$$

The procedure is repeated until τ satisfies the condition $\dot{x}_1(\tau) = 0$. The stuck position of the first mass with the converged τ becomes

$$x_1(\tau) = \theta(\tau) - \frac{m_2}{m1+m2}\psi_0 \tag{46}$$

This value should agree to the value of x_{1s} chosen to determine η_0, $\dot{\eta}_0$ and A_3 shown in Equation 31. If $x_1(\tau)$ results in a larger value than x_{1s}, the larger value of x_{1s} should be selected. Therefore, the new x_{1s} can be updated by the following relationship.

$$x_{1s,new} = x_{1s,old} + K[x_1(\tau) - x_{1s,old}] \tag{47}$$

where, K is the update gain. The procedure of designing a controller discussed so far is summarized in Figure 7. The controller design so far assumes that the first mass stays stuck for $\tau \le t \le T_3$. However, if the spring force is large enough to move the first mass, the spring force should be compensated to stay stuck. Therefore, the control input should be modified such that

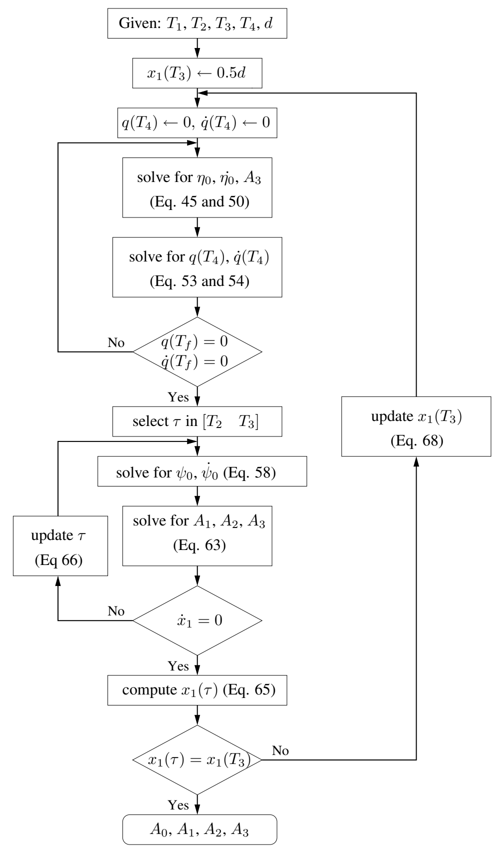


Fig. 7 Algorithm for the Controller Design under Stiction

$$u = -k\left[\psi_0\cos\omega_s(t-\tau) - \dot{\psi}_0\frac{\sin\omega_s(t-\tau)}{\omega_s}\right] \qquad \tau < t < T_3 \tag{48}$$

The corresponding new control profile is shown in Figure 8. It is also possible to compensate the spring force with a constant input force. Define u_{max} and u_{min} as maximum and minimum input force for $\tau \le t \le T_3$. The constant input can be used for spring

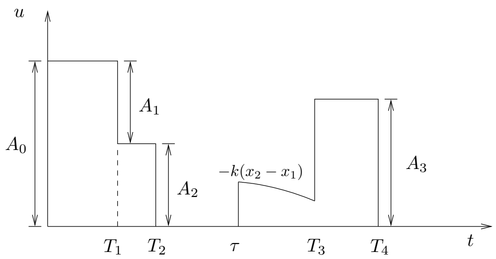


Fig. 8 Input Profile with Spring Force Compensation

force compensation such that:

$$u = u_{max} \qquad \text{if } u_{max} - u_{min} < f_s(\tau \leq t \leq T_3) \tag{49}$$

NUMERICAL EXAMPLE

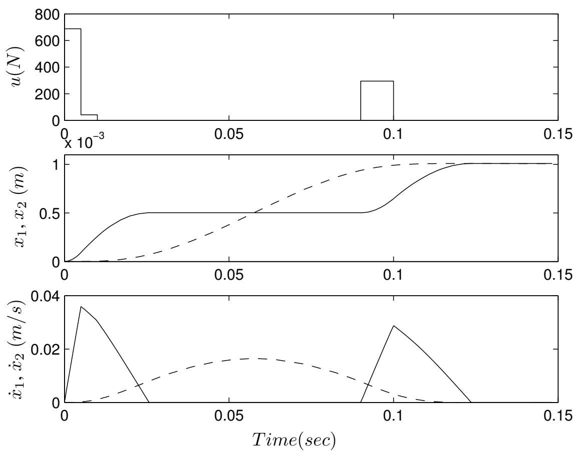


Fig. 9 Control Input and Responses($d = 0.001m$)

The same example problem in Section is used for the new control profile. The boundary conditions for the problem are

$$\begin{array}{ll} x_1(0) = x_2(0) = 0 & x_1(T_f) = x_2(T_f) = 0.001 \\ \dot{x}_1(0) = \dot{x}_2(0) = 0 & \dot{x}_1(T_f) = \dot{x}_2(T_f) = 0 \end{array} \tag{50}$$

The pulse widths are selected such that $T_1 = 0.005$ sec, $T_2 = 0.01$ sec, $T_3 = 0.09$ sec and $T_4 = 0.1$ sec. The resulting control input and responses of the system are plotted in Figure 9. It is shown from the response plot that the first mass stuck position is very close to the half way of the desired final position. The first mass stays stuck without compensating for the spring force because the spring force is smaller that the static friction value. If the spring force is greater than the static friction during the stiction, the control input should be modified to include spring force compensation.

CONCLUSION

In this work, design techniques for pulse amplitude modulated controllers are presented. A three pulse controllers with user selected pulse width initiates the motion of the maneuvering system towards its final position and then exploits the friction force to bring the coasting system to rest. This assumes unidirectional frictional force which requires that the mass which is subject to friction has a velocity profile which does not change sign. Variations in the pulse amplitude as a function of displacement and modal frequency is studied and it is noted that for specific frequencies, the proposed technique results in infeasible solutions. This problem can be addressed by selecting different pulse widths. Next, a modification of the three pulse control profile is proposed which can account for stiction and velocity reversals. This technique requires the spring force to be compensated if it is greater than the static friction.

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